DEL PEZZO SURFACES AND GROUP ACTIONS ON ALGEBRAIC VARIETIES IN MAGMA

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1. ALPHA-INVARIANT OF TIAN

The α_G–invariant of Tian is a numerical invariant of a pair (V, G), where V is a smooth Fano variety (see Section 4) and G is a finite group acting bi–regularly on V. This invariant coming from Kähler Geometry is identical to the G-invariant global log canonical threshold, $\mathrm{lct}(V, G)$, in Algebraic Geometry ([CS08, Appendix A]). In Algebraic Geometry, singularities are a natural part of the minimal model program (MMP) in dimensions greater than two; the MMP is an (almost complete) method to decide in which birational class any given variety lies. The local log canonical threshold is a tool to classify the worst singularities and crops up in many of the inductive proofs in the MMP. Furthermore, there are many connections between it and the classification of certain quotient singularities (see [Sho00, MP99, CS09]).

Below (in Section 2) we define this invariant and give examples of its calculation. We also present some tools developed within the Magma computer algebra system ([BCP97]) during the author's visit to the Computer Algebra Group at the University of Sydney. First though, we'd like to present some interesting and seemingly disparate applications of these α–invariants / log canonical thresholds.

Conjugacy in Cremona groups. The Cremona group $\mathrm{Cr}_n(\Bbbk)$ is the group of birational automorphisms of projective n–space, $\mathbb{P}^n$, over a field $\Bbbk$. The Cremona group of the projective line is isomorphic to $\mathrm{PGL}_2(\Bbbk)$. The Cremona group of the plane, $\mathrm{Cr}_2(\Bbbk)$, is already large and complex. In higher dimensions, our understanding is even poorer. One step towards a holistic understanding of the structure of $\mathrm{Cr}_n(\Bbbk)$ is the classification of its conjugacy classes. A modern approach to this, initiated by Iskovskikh and Manin (see e.g. [Man67, Isk80]), is to consider rational G-varieties and G-equivariant maps between them. Indeed, there is a natural correspondence between G-equivariant birational isomorphism classes of rational G-varieties and conjugacy classes of subgroups of $\mathrm{Cr}_n(\Bbbk)$ isomorphic to G (see e.g. [Dol10]).

The following simple observation provides one opening to study conjugacy in higher dimensional Cremona groups. Namely, consider a rational variety X over $\Bbbk$ of dimension n; that is a variety where there exists a birational map over $\Bbbk$, $\psi : X \to \mathbb{P}^n$. Then for any group G acting bi–regularly on X, the group $\psi G \psi^{-1}$ is clearly a subgroup of $\mathrm{Cr}_n(\Bbbk)$. If in addition ψ, or any similar map to $\mathbb{P}^n$ is non–G–equivariant (i.e. X is non–G–rational), then it follows that $\psi G \psi^{-1}$ is not a sub–group of the automorphisms of $\mathbb{P}^n$. Hence G is not conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ in $\mathrm{Cr}_n(\Bbbk)$. Already this simple observation is powerful, as we see in the following example.

Example 1. Let X be a smooth Fano variety of dimension two such that the self–intersection of the anti–canonical divisor is five (i.e. a del Pezzo surface of degree five; see Section 4 below). Then X is the blowup of four points $P_1, \ldots, P_4$ in $\mathbb{P}^2$ and there are ten (-1)–curves on X (curves whose self–intersection is -1): the four exceptional curves E_i coming from the blown up points P_i and six others coming from the strict transforms of the lines on $\mathbb{P}^2$ passing through two of the P_i. As any four points in $\mathbb{P}^2$ can be mapped projectively to any other four points it follows that the surface X is unique up to isomorphism. The five sets $F_k = \{E_k, D_{ij} \mid i, j \neq k\}$ (for $k \in \{1, \ldots, 4\}$) and $F_5 = \{E_1, \ldots, E_4\}$ contain (-1)–curves that do not intersect one–another, thus any of these sets defines a map from X to $\mathbb{P}^2$ given by their contraction to points and this exhausts all possible birational maps to the projective plane. The symmetric group $\mathbb{S}_5$ acts on these five sets in the standard way and it is not hard to see that the entire automorphism group of X is given in this way.

Consider the alternating sub–group $G = \mathbb{A}_5$ of $\mathbb{S}_5$. Under the action of G on S all the (-1)–curves E_k, D_{ij} are in one orbit, which shows that there cannot be any G–equivariant maps to $\mathbb{P}^2$ and hence S is non–G–rational. Also, $G' = \mathbb{A}_5$ is well know to have a bi–regular action on $\mathbb{P}^2$. From our simple observation then we can conclude that G is not conjugate to G' in $\mathrm{Cr}_2(\Bbbk)$. In fact, there are three conjugacy classes for $\mathbb{A}_5$ in $\mathrm{Cr}_2(\Bbbk)$, the third being realised as an action on $\mathbb{P}^1 \times \mathbb{P}^1$.

With knowledge of Tian's α_G–invariant and the action of the group G on some given Fano varieties we can make use of a Theorem of Cheltsov–Pukhlikov together with the above observation to study conjugacy in higher rank Cremona groups, we present only a corollary to their theorem here and leave the interested reader to consult [Che09]. Suppose that for $i = 1, \ldots, k$; S_i is a smooth del Pezzo surface and that G_i is a finite group acting bi–regularly on S_i. Then if for all G_i–orbits Σ_i on S_i we have both $|\Sigma_i| \geqslant K_{S_i}^2 = \{1, \ldots, 9\}$ and $\alpha_{G_i}(S_i) = \mathrm{lct}(S_i, G_i) \geqslant 1$ it follows that $S = S_1 \times \ldots \times S_k$ is non–G–rational, where $G = G_1 \times \ldots \times G_k$. In our previous example, it is known (see [Spr77, YY93]) that $|\Sigma| \geqslant 6$ and proved in [Che08] that $\alpha_{\mathbb{A}_5}(X) = \mathrm{lct}(X, \mathbb{A}_5) = 2$.

Kähler Geometry. The existence of a G–invariant Kähler metric on a smooth Fano manifold V whose Ricci curvature is proportional to the metric tensor, that is a Kähler–Einstein metric, is a well studied problem in Kähler geometry. The only known sufficient condition for variety the V to admit such a metric is that of Tian, Siu, Nadel, Kollàr and Demailly,

$$\alpha_G(V) > \frac{\dim(V)}{\dim(V)+1}.$$

The problem of existence of (non-invariant) Kähler–Einstein metrics on smooth Fano surfaces (del Pezzo surfaces) was completely solved by Tian. Namely, a del Pezzo surface admits a Kähler–Einstein metric whenever it is not the blowup of one or two points of the projective plane, or equivalently whenever its full automorphism group is reductive.

The α_G-invariant is also significant in determining the convergence of the Kähler–Ricci flow and Kähler–Ricci iterations. For a Kähler form ω in the first Chern class, the Kähler–Ricci iteration is defined as

$$\omega_n = \mathrm{Ricci}(\omega_{n+1}); \quad \omega_0 = \omega.$$

Phong, Sturm and Rubinstein used deep estimates of Perelman to show that if the α_G–invariant is greater than one, then the Kähler–Ricci iteration converges exponentially fast to the Kähler form associated to a Kähler-Einstein metric in the $C^\infty(V)$-topology.

2. Background on Global Log Canonical Thresholds

Before describing the tools implemented in Magma let us first explore the global log canonical threshold of a Fano variety, the algebro-geometric equivalent of Tian's α–invariant and how to calculate it. As we mentioned above, singularities arise naturally in the MMP and we quickly require methods to measure their severity; these tools are the discrepancy and the log canonical threshold (lct), more background details can be found in [Wil10].

Measuring singularities: the discrepancy. The discrepancy of a (log) pair (V,Δ) where V is a normal variety and Δ a ($\mathbb{Q}$–Cartier) divisor on V is a numerical invariant. Its calculation involves resolving the singularities of the pair, keeping track of the pullback of the canonical divisor of V and comparing the results upstairs on the resolution. Let us illustrate this with an example.

Example 2. Consider the affine curve C on the plane $\mathbb{A}^2$ with coordinates x, y given by the zeros of $(x+y)(x-y)$. This is of course the union of two lines L_1, L_2 meeting transversely at the origin. To calculate the discrepancy of $(\mathbb{A}^2, C)$ we should blowup the origin $O \in \mathbb{A}^2$ to resolve the nodal singularity of C. Let $\pi : \mathrm{Bl} \to \mathbb{A}^2$ be this blowup, with exceptional curve $E \simeq \mathbb{P}^1$ ($\pi(E) = O$). On one chart of the blowup the pullback of is given by $x \mapsto uv, y \mapsto v$, where u, v are the new coordinates and E is given by the equation $v = 0$. Then the pullback of C is given by the zeros of $(uv+v)(uv-v) = v^2(u+1)(u-1)$, which written additively is $2E+\overline{L_1}+\overline{L_2}$, where bar denotes strict transform. Next we keep track of the canonical divisor $K_{\mathbb{A}^2}$, of course we have $\pi^*(K_{\mathbb{A}^2}) = K_{\mathrm{Bl}} + kE$ as the blowup is an isomorphism away from E and it is not too difficult to see that $k = 1$ from the Riemann–Roch formula. Lastly, we compare these upstairs on Bl.

$$K_{\mathrm{Bl}} + \overline{C} = \pi^*(K_{\mathbb{A}^2} + C) - E.$$

The coefficient of E, -1, we call the discrepancy of the pair $(\mathbb{A}^2, C)$.

In general, for a normal variety V such that $V + \Delta$ is $\mathbb{Q}$–Cartier (so that intersections / pullbacks / etc. are well defined) where Δ is an effective $\mathbb{Q}$–Cartier divisor on V and $f : U \to V$ is a birational morphism with exceptional divisors E_i we can write

$$K_U + \overline{\Delta} = f^*(K_V + \Delta) + \sum_i a(V,\Delta;E_i)E_i.$$

Then the discrepancy of the pair (V,Δ) is the number

$$\mathrm{discrep}(V,\Delta) = \inf_E\{a(V,\Delta;E) | E \text{ exceptional over } V\},$$

where exceptional means that $\mathrm{codim}(f(E)) \geqslant 2$ and $f(E) = \emptyset$. Note that the definition is independent of of f, indeed we only need to examine the 'log resolution', that is the resolution of the pair where the support of the strict transform of Δ and the exceptional locus can be locally given by the equation xy (i.e. has simple normal crossings) and U is smooth. Then we say that the pair (V,Δ) is terminal, canonical, (purely) log terminal (plt), log canonical (lc) if $\mathrm{discrep}(V,\Delta)$ is $> 0, \geqslant 0, > -1, \geqslant -1$, respectively.

Example 3. For $\Delta = 0$, a curve is plt if and only if it is smooth and lc if and only if it has at worst nodal singularities. A surface is terminal if and only if it is smooth and canonical precisely when it has at worst Du Val (or ADE) singularities.

Further details can be found in [Kol97, KM98, KSC04].

Measuring singularities: local lct. The discrepancy is not refined enough to discern the severity of all singularities however, as we'll see in the following example if $\mathrm{discrep}(V,\Delta;E) > -1$ then $\mathrm{discrep}(V,\Delta;E) \to -\infty$.

Example 4. Let's modify our previous curve $C \subseteq \mathbb{A}^2$ to be now given by the zeros of $x(x+y)(x-y)$. After one blowup at the origin the pullback of C is given by $\pi^*(C) = \bar{C} + \mathrm{multi}_O(C)E = \bar{C} + 3E$ and so $\mathrm{discrep}(\mathbb{A}^2, C; E) = -2$. However, now on successively iterating blowups at a smooth point of $E \setminus \bar{C}$ we find that $\mathrm{discrep}(\mathbb{A}^2, C; E_2) = -5, \mathrm{discrep}(\mathbb{A}^2, C; E_3) = -21, \ldots$ where E_2 and E_3 are the exceptional curves of the second and third blowups as the multiplicities of the new exceptional curves quickly explode. It is not hard to see that $\mathrm{discrep}(\mathbb{A}^2, C; E_i) \xrightarrow[i\to\infty]{} -\infty$.

To rectify our inability to measure singularities with this behaviour, we observe that $(\mathbb{A}^2, C)$ is not lc whereas $(\mathbb{A}^2, 0\cdot C)$ is. Thus there exists a maximal threshold λ such that $(\mathbb{A}^2, \lambda\cdot C)$ is lc. Hence, with V and Δ as above, we define the (local) log canonical threshold at a point $P \in V$ with respect to a divisor Δ to be the number

$$\mathrm{lct}_P(V,\Delta) = \sup\{\lambda \in \mathbb{Q} | (V, \lambda\Delta) \text{ lc at } P\} \in [0,1].$$

For our previous example, we wish $(1-3\lambda) \geqslant -1$, that is $\lambda \leqslant \frac{2}{3}$. Hence, $\mathrm{lct}_O(\mathbb{A}^2, C) = \frac{2}{3}$.

It is worth noting here that in Magma there exists the function `ResolutionGraph()` that returns the log resolution of an affine curve at the origin, whose algorithm works via Newton polynomials (see [KSC04] for details on this method).

Measuring singularities: global lct. Back in our general setting we may define the lct of a log pair globally by setting

$$\begin{aligned}\mathrm{lct}(V,\Delta) &= \inf\{\mathrm{lct}_P(V,\Delta) | P \in V\}\\ &= \sup\{\lambda \in \mathbb{Q} | (V,\lambda\Delta) \text{ lc }\}.\end{aligned}$$

Expanding on our previous example again we have the following.

Example 5 ([CPS08, Example 1.1.3])**.** Let D be a cubic curve on the projective plane $\mathbb{P}^2$. Then

$$\mathrm{lct}\big(\mathbb{P}^2, D\big) = \begin{cases} 1 & \text{if } D \text{ is a smooth curve,}\\ 1 & \text{if } D \text{ is a curve with ordinary double points,}\\ 5/6 & \text{if } D \text{ is a curve with one cuspidal point,}\\ 3/4 & \text{if } D \text{ consists of a conic and a line that are tangent,}\\ 2/3 & \text{if } D \text{ consists of three lines intersecting at one point,}\\ 1/2 & \text{if } \mathrm{Supp}\big(D\big) \text{ consists of two lines,}\\ 1/3 & \text{if } \mathrm{Supp}\big(D\big) \text{ consists of one line.}\end{cases}$$

Calculating the global lct on Fano varieties. On Fano varieties we have an obvious choice of divisor to choose for our global lct definition, the anti-canonical divisor. Moreover, to mimic Tian's original α–invariant definition (see [CS08, Appendix A]) we also consider the bi–regular action of a finite[1] group G on our Fano variety V and define the *G-invariant global log canonical threshold* of $\big(V,G\big)$ to be the number

$$\begin{aligned}\mathrm{lct}\big(V,G\big) &= \inf\Big\{\mathrm{lct}\big(V,\Delta\big)\Big| \Delta \text{ is an effective } G\text{-invariant } \mathbb{Q}\text{-divisor on } V \text{ such that } \Delta \sim_{\mathbb{Q}} -K_V\Big\}\\ &= \sup\Big\{\lambda \in \mathbb{Q}\Big| \text{the log pair } \big(V,\lambda\Delta\big) \text{ is lc for all } G\text{-invariant } \mathbb{Q}\text{-divisors } 0 \leqslant \Delta \equiv -K_V\Big\}.\end{aligned}$$

The above definition of the log canonical threshold is, in practise, difficult to work with. To calculate these thresholds, as we'll see below, we look in the pluri-anti-canonical linear systems for the 'worst' G-invariant divisors (i.e. those with the smallest log canonical threshold) and prove that they realise the global G-invariant log canonical threshold. It makes sense then to split the definition with an intermediate definition as

$$\mathrm{lct}_m\big(V,G\big) = \sup\Big\{\lambda \in \mathbb{Q}\Big| \text{ the log pair } \big(V, \frac{\lambda}{m}\Delta\big) \text{ is lc for all } G\text{–invariant divisors } \Delta \in |-mK_V|\Big\}.$$

Then

$$\mathrm{lct}\big(V,G\big) = \inf\Big\{\mathrm{lct}_m\big(V,G\big)\Big| m \in \mathbb{N}\Big\} \geqslant 0.$$

Note that when $|-mK_V|$ contains no G-invariant divisors, $\mathrm{lct}_m\big(V,G\big)$ is defined to be $+\infty$. Write lct(V), etc. when G is trivial.

To date, no Fano varieties with non-rational global log canonical thresholds have been found. We expect this property to hold for all global log canonical thresholds. Furthermore, we expect that the global log canonical threshold is realised by a divisor in one of the pluri-anti-canonical linear systems (see [CPS08] and [Tia90]). These divisors, numerically equivalent to the anti-canonical divisor, whose log canonical threshold realises the global log canonical threshold are called *wild tigers.* In this colourful language of Keel-MacKernan ([KM99]), we say that

[1] We can consider also the action of a compact group by altering the definition of the lct to look not at G–invariant divisors, but G–invariant linear systems.

the calculation of global log canonical thresholds is, in part, the *hunt for wild tigers* (cf. [CP02]). In [Wil10] the following conjecture is confirmed for the case where (X,G) is a smooth del Pezzo G-surface and G is finite.

Conjecture 6. *For a Fano variety V, let G be a finite subgroup of* $\mathrm{Aut}(V)$. *Then there exists an effective G-invariant $\mathbb{Q}$-divisor, $\Delta \sim_{\mathbb{Q}} -K_V$ such that* $\mathrm{lct}(V,G) = \mathrm{lct}(V,\Delta) \in \mathbb{Q}$.

We conclude this section by exemplifying the above definitions, showing that $\mathrm{lct}(\mathbb{P}^2) = 1/3$.

Example 7. We first examine the linear systems $\mathscr{L}_m = |-mK_{\mathbb{P}^2}|$to find the smallest m such that $\mathscr{L}_m$ contains G–invariant divisors (of course G is trivial in our case). Then, from Example 5 we see that as $-K_{\mathbb{P}^2} \sim 3L$ where L is a line on $\mathbb{P}^2$, $\mathscr{L}_1$ is non-empty and it follows that $\mathrm{lct}_1(\mathbb{P}^2) = 1/3$. Next we must prove a special case of Conjecture 6, which in general is difficult however with a few tricks and the use of Nadel–Shokurov Vanishing (see e.g. [Wil10], Vanya paper) can be achieved in many cases. For our case we may do the following: suppose that there exists $\lambda in \mathbb{Q}$ such that $\mathrm{lct}(\mathbb{P}^2) < \lambda < \mathrm{lct}_1(\mathbb{P}^2) = 1/3$. Then, from the definition, there is an effective $\mathbb{Q}$–divisor Δ numerically equivalent to $-K_{\mathbb{P}^2}$ such that $(\mathbb{P}^2, \lambda\Delta)$ is not lc. Write $\Delta = 3D$ for some divisor D, then also $(\mathbb{P}^2, D)$ is not lc. Hence there exists a point $P \in \mathrm{Supp}(D)$ such that $\mathrm{multi}_P(D) > 1$. However, for a general line $L \ni P$ we have

$$1 = L \cdot D \geqslant (\mathrm{multi}_P(L))(\mathrm{multi}_P(D)) > 1.$$

3. Aiding the calculation of α–invariants in Magma

We continue now by looking at some newly implemented functions in Magma. These tools now available are to aid the computation of global group–invariant log canonical thresholds, namely: finding the group–invariant parts of Riemann–Roch spaces, which for us corresponds to finding invariant curves and linear systems in the pluri–anti–canonical linear system of some del Pezzo surface; calculating local and global log canonical thresholds of curves; and computing some subgroups of the full automorphism group on cubic surfaces by finding their Eckardt points (points where three lines contained in the surface meet at a single point).

Actions on Riemann–Roch spaces. For a finite group G, given the equations of a variety $V \subseteq \mathbb{P}^n$ and an explicit bi–regular action of G on V (i.e. generators acting on the homogeneous coordinates of $\mathbb{P}^n$) we have an explicit action of G on the Riemann–Roch space of D, $H^0(V,D)$ (i.e. some matrices corresponding to the generators) for some effective divisor D. We can determine if this action on $H^0(V,D)$ is irreducible or not, and if not, how it splits. The invariant sub-spaces correspond to G–invariant curves C_i (or G–invariant (sub–)linear systems of curves) in the complete linear system $|D|$. To this end we implemented the following functions:

- `InvariantPolynomials()`
  `Input:` Projective Space $\mathbb{P}$, sequence S of generating matrices for the action of a group G on $\mathbb{P}$, integer d.
  `Output:` Equations of G-invariant divisors and generating divisors of G-invariant linear systems in $\mathrm{Proj}(H^0(P,\mathscr{O}_{\mathbb{P}}(d))) = |-dK_{\mathbb{P}}|$.
  `Algorithm:`
  * Form the matrix group MG corresponding to the generators in S;
  * construct the G–module obtained by the action of MG on the homogeneous polynomials of degree d in the coordinate ring of $\mathbb{P}$;
  * find the indecomposable summands of the G–module and for the one-dimensional summands, the corresponding equations of the divisors on $\mathbb{P}$;
  * calculate the eigenvalues of each of these one dimensional summands and determine which, if any, agree across all the generators of MG;
  * return the G–invariant divisors and group together any divisors compatible under the action (these may form G-invariant linear systems).
- `ActionOnDivisor()`
  `Input:` Map $g: V \to V$ of a variety V, Divisor D on V.
  `Output:` $g(D)$.
- `ActionOfElementOnRiemannRoch()`
  `Input:` Map $g: V \to V$ of a variety V, Divisor D on V.
  `Output:` Action of g on $H^0(V,D)$, $\gamma: H^0(V,D) \to H^0(V,D)$.

Log canonical thresholds of curves. As described in Section 2, we can calculate the lct of a variety with the knowledge of its log resolution. Building on the function `ResolutionGraph()` we implemented the following.

- `LogCanonicalThresholdAtOrigin()`
  `Input:` affine curve C
  `Output:` $\mathrm{lct}_O(C)$
  `Algorithm:`
  * If C is reduced, then we use `ResolutionGraph()`
  * If C is non–reduced, then we find the multiplicity of the components of C passing through the origin using either `RemoveHypersurface()` (which factors out the equation of the component) if the curve is a hypersurface, or by using Colon ideals if not.
- `LogCanonicalThreshold()`

Input: curve C (affine or projective), point P
Output: $\mathrm{lct}_P(C)$
Algorithm: we work on an affine patch, translate the point P to the origin and use `LogCanonicalThresholdAtOrigin()`.

- `LogCanonicalThreshold()`
  Input: curve C (affine or projective)
  Output: $\mathrm{lct}(C)$ (over base field of C)
  Algorithm: we compute the singular points $\{P_i | i \in I\}$ of C over the base field using `SingularPoints()`, run `LogCanonicalThreshold(C,P_i)` for each $i \in I$ and take the minimum possible value.
- `LogCanonicalThresholdOverExtension()`
  Input: curve C (affine or projective)
  Output: $\mathrm{lct}(C)$ (calculated over all singular defined over extensions)
  Algorithm: uses `PointsOverSplittingField()` and `LogCanonicalThreshold()`.

Calculating Eckardt involutions for cubic surfaces. For a cubic surface (i.e. a del Pezzo surface of degree three), there is a well known correspondence between Eckardt points and involutions of the surface. An Eckardt point P of a cubic surface S is a point where three lines lying on the surface meet concurrently. Suppose S is smooth, blowing up the point P yields a degree two del Pezzo surface T that is a double cover of the projective plane ramified in a smooth degree four curve (see Section 4). The (Geiser) involution that interchanges the sheets of the double cover can be composed with the blowdown to S to given an involution associated to the point P. As part of the del Pezzo package described below we implemented a function to find all Eckardt points of a given cubic surface and using this we added a function to return the associated involutions.

4. DEL PEZZO SURFACES

A variety with ample anti–canonical divisor (that is, some multiple of it defines an embedding) is called Fano, well known examples are projective n–space, varieties of degree m in $\mathbb{P}^m$ and cubic surfaces. A dimension two Fano variety is called a del Pezzo surface, which is either smooth or has at worst Du Val singularities. There is a well known classification (see e.g. [KSC04]); a smooth del Pezzo surface is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the blowup of $\mathbb{P}^2$ in $r (< 9)$ points in general position. Here general position means that no three lie on a line, no six lie concurrently on a conic and no eight lie on a cubic where one is a double point. The degree d of a del Pezzo surface is the self–intersection number of the anti–canonical divisor, $d = K_S^2 = 9 - r$. They have standard anti–canonical embeddings given when $r < 7$. For $r = 7, 8$ the anti–canonical divisor is not very ample (that is, does not define an embedding). For $r = 7$ we have that $-2K_S$ is very ample and this yields a pluri–anti–canonical embedding in the weighted projective spaces $\mathbb{P}(1,1,1,2)$ which is a double cover of $\mathbb{P}^2$. Similarly, for $r = 8$, $-3K_S$ is very ample, giving an embedding in $\mathbb{P}(1,1,2,3)$ that is a double cover of the quartic cone $\mathbb{P}(1,1,2)$.

Together with Gavin Brown, Martin Bright and Steve Donnelly we implemented the following functions.

- `IsDelPezzo()`
  Input: scheme Y
  Output: boolean b and (pluri–)anti–canonical embedding φ
  Algorithm: the function makes the following checks
    * is the dimension of Y two?
    * is $K_Y^2 \in \{1, \dots, 9\}$?
    * is the dimension the linear system $|-K_Y| = K_Y^2$?
  If the answer to all the above is true then it calculates φ and returns true.
- `PointsInGeneralPosition()`
  Input: list of points L
  Output: boolean (optionally returns the offending lines / conics / etc.)
  Algorithm: calculates the set of all lines, conics and cubics with a double point in L and compares their number to the maximal possible.
- `DelPezzoSurface()`
  Input: list of points L
  Output: del Pezzo surface S
  Algorithm: checks the points of L are in general position using `PointsInGeneralPosition()`. For $|L| < 7$, forms the linear system $|-K_{\mathbb{P}^2} - P_1 - \dots - P_r|$ i.e. the linear system of cubics passing through the points of L and returns the image of the associated map. For $|L| = 7$, computes $|-K_{\mathbb{P}^2} - P_1 - \dots - P_7|$ as before and also finds a section s of $|-2K_{\mathbb{P}^2} - P_1 - \dots - P_7|$ not given by sections of $|-K_{\mathbb{P}^2} - P_1 - \dots - P_r|$ and returns the image of the map given by three sections of $|-K_{\mathbb{P}^2} - P_1 - \dots - P_7|$ and s. For $|L| = 8$, as for $|L| = 7$ but we find two sections in $|-K_{\mathbb{P}^2} - P_1 - \dots - P_8|$, one in $|-2K_{\mathbb{P}^2} - P_1 - \dots - P_8|$ and one in $|-3K_{\mathbb{P}^2} - P_1 - \dots - P_8|$.
- `EckardtPoints()`
  Input: cubic surface S
  Output: list of Eckardt points L
  Algorithm: computes the corresponding Hessian surface H using `HessianMatrix()` and returns points lying on both H and S.

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