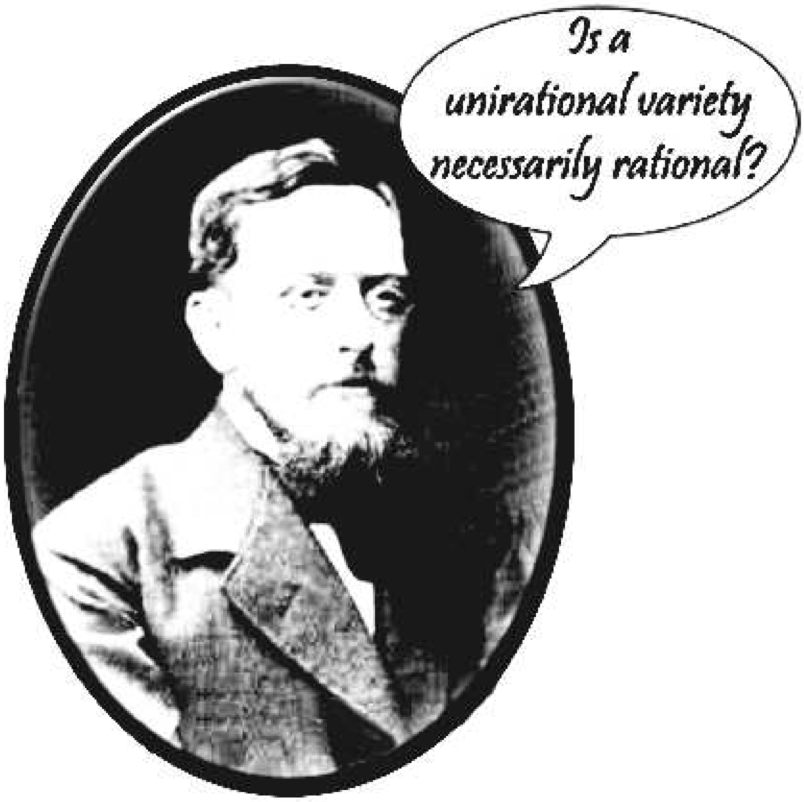
On the Lüroth Problem



Andrew Jowan Wilson

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Dr. Ivan Cheltsov

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Introduction

One of the main problems in algebraic geometry is the classification problem; classify, up to isomorphism, all the algebraic varieties. The first step towards this goal is to solve the weaker problem; classify, up to birational equivalence, all the algebraic varieties. The natural beginning to this is to gain an understanding of the simplest varieties – projective spaces. Rational varieties are their closest relatives and it is these, together with unirational varieties, that we consider here.

Recall, that a *rational variety* is a variety birationally equivalent to a projective space. That is, there exist mutually inverse rational maps with coefficients in a field k, from the variety to a projective space and back. Equivalently, as each of the rational maps is dominant (i.e. surjective on open subsets) the pullback defines an inclusion of the function fields. Hence a variety is rational if, and only if, its function field[1] is isomorphic to the field of rational functions in n variables $k(x_1, \dots, x_n)$, i.e., the function field of a projective space $\mathbb{P}^n$.

A *unirational variety* is one covered by a rational variety. That is to say that there is a dominant rational map, with coefficients in a field k, from a projective space to the variety. Again, the pullback will define an inclusion of the function field of the variety in the field of rational functions in n variables, where n is the dimension of the variety. Thus, a variety is unirational if, and only

[1]The function field of a variety X is defined to be the field of all rational functions on X.

if, there is some rational function field such that the field of rational functions of the variety is contained inside.

One of the first things to note, beyond the trivial observation that any rational variety is unirational, is that any variety birationally equivalent to a unirational variety is itself unirational. Thus, unirational varieties make up several birational equivalence classes, with rational varieties one of these. A natural question to ask is; are there really several distinct equivalence classes corresponding to the unirational varieties, or just one – corresponding to the rational varieties? Put another way; are all unirational varieties rational? Put in yet another way; are all non-trivial subfields of the field of rational functions in fact isomorphic to the field of rational functions?

This is the question Lüroth posed in 1861, which became known as the 'Lüroth Problem'. It went on to inspire many great mathematicians and is seed for much research. The answer to this question is of course negative in general, and can be summed up in the following table.[2]

dim	any field k	alg. closed k of char. 0 (i.e. $\mathbb{C}$)	alg. closed k of char. p s.t. $k(x_1,\ldots,x_n)/k$ is a separable field ext.
1	+	+	+
2	-	+	+
≥ 3	-	-	-

Where '+' indicates that the answer to the question is affirmative and '-' indicates that the answer is not necessarily affirmative. We shall go through these results in detail in what follows.

We focus, in each of the chapters, on the Lüroth problem in a different dimension. Following this introduction, chapter one discusses the case of curves; this is Lüroth's theorem, from which we get the affirmative answer indicated in the table above. In chapter two we discuss the case of unirational surfaces and examine some rationality criterion, namely Noether's lemma and Castelnuovo's rationality criterion. After which we look at our fist counterexamples to the Lüroth problem. The third chapter looks at higher dimensions. More specifically, we examine three-folds and see what obstructions there are for unirational three-folds to be rational in general. Finally, we include an appendix of some well known results that we assume in the course of this survey.

[2] Inspired by a table on the first page of [Katsura].

1. Lüroth's Theorem

Lüroth's theorem, as is often the case in the domain of algebraic geometry, may be stated in the language of algebra or of geometry. Firstly, we shall state it in algebraic terms and then reformulate the theorem into the language of geometry.

Theorem 1. *(Lüroth) Suppose we have an inclusion of fields $k \subset L \subseteq k(t)$, where $k \neq L$ and $k(t)$ is the field of rational functions in one variable t. Then L is isomorphic to $k(t)$.*

What does this mean geometrically? Suppose that $X \subseteq \mathbb{P}^n$ is a unirational variety of dimension one. Then there exists a dominant rational map $\varphi : \mathbb{P}^1 \dashrightarrow X$, that is to say that the image of φ is dense in X. Thus, the pullback map defines an isomorphic inclusion

$$\varphi^* : k(X) \hookrightarrow k(\mathbb{P}^1)$$

Recall that for any (quasi-)projective variety Y and any open subset $U \subseteq Y$ we have $k(U) = k(Y)$. Here, $\mathbb{A}^1 \subseteq \mathbb{P}^1$ is open and so $k(\mathbb{P}^1) = k(\mathbb{A}^1) = k(\tau)$ for some variable τ.

So it is then that we find ourselves in the situation of Lüroth's theorem; if we assume that X is not a point, so that $k(X) \neq k$, then we conclude that $k(X)$ is isomorphic to $k(\mathbb{P}^1)$. Hence the condition that a one dimensional variety be unirational automatically qualifies it as being rational.

In keeping with the spirit of viewing results from both an algebraic and geometric point-of-view, we give below a proof of Lüroth's theorem from both sides; 1.1 for the geometric and 1.2 for the algebraic.

1.1. **Proof of Lüroth's theorem via Riemann-Hurwitz and Riemann-Roch**

Let C be a unirational plane curve. We desire to show that it must, in fact, be rational. By definition, there exists a dominant rational map $\varphi : \mathbb{P}^1 \dashrightarrow C$. We use the Riemann-Hurwitz formula (see Appendix 4.3) to obtain

$$\chi(\mathbb{P}^1) = N\chi(C) - ram_\varphi$$

where N is the degree of φ.

Thus,

$$2 - 2g_{\mathbb{P}^1} = N(2 - 2g_C) - ram_\varphi$$

that is,

$$g_C = 1 - \frac{1}{N} - \frac{ram_\varphi}{N}$$

Since the genus of any curve is a non-negative number and as $ram_\varphi \geq 0$ we have $1 - \frac{1}{N} - \frac{ram_\varphi}{N} < 1$. In conclusion, C is of genus zero and hence a rational curve. □

1.2. **An algebraic proof of Lüroth's theorem**

We now sketch an elementary algebraic proof of Lüroth's theorem. For details see [Waerden] Section 63.

Considering any element $\lambda \in L \setminus k$, we observe that t is an algebraic element of $k(\lambda)$ and so an algebraic member of L. Next, examine the polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in L[z]$$

where the a_i are rational functions in x. Multiplying through by the lowest common denominator yields polynomials and we may write

$$f(x,z) = b_0(x)z^n + b_1(x)z^{n-1} + \cdots + b_n(x)$$

Let the degree of f with respect to x be m.

Note that not all the coefficients $a_i = \frac{b_i}{b_0}$ in $f(x)$ can be independent of x, since that would imply that x is algebraic with respect to k. Thus at least one of the terms $a_i = \theta$ must be dependant on x.

To complete the proof, one uses elementary field extension properties to show that (i) $m = n$, and (ii) θ, as a function of x, is of degree m. It follows that

$$[k(x) : k(\theta)] = m = [k(x) : L]$$

and as $L \supseteq k(\theta)$, we have $[L : k(\theta)] = 1$. Therefore

$$L = k(\theta) \cong k(t)$$

by a change of variables.

2. The Lüroth Problem in Dimension Two

Surfaces provide us with the first counterexamples to the Lüroth problem. Here we shall explore some invariants that allow us to decide when a surface is rational, much like the genus of a curve in the one-dimensional case. This will lead us naturally to a Lüroth Theorem for surfaces, with some restrictions on the underlying field. We will then go on to show some explicit examples of surfaces that are unirational, but not rational. Thus, demonstrating the necessity of the restrictions on the Lüroth theorem for surfaces.

2.1. **Preliminaries**

Before starting, let us recall some basic facts we shall need in this exploration (see [G-H], for example); Hodge numbers are symmetric, $h^{p,q}(X) = h^{q,p}(X)$ for some variety X and $h^{p,q}(X)$ is defined by $h^{p,q}(X) = h^q(\Omega^p[X])$. The Hodge numbers $h^{p,0}$ are birational invariants; $h^{p,0}(X) = h^0(\Omega^p[X]) = h^p(\Omega^0[X]) = h^p(\mathcal{O}_X)$. Many of these invariants were discovered before the modern theory and some picked up names along the way, which we give now; the *geometrical genus* of a n-fold X is the number $p_g(X) = h^{n,0}(X) = h^0(\Omega^n[X]) = h^0(K_X)$, when $n = 1$ and X is a smooth curve this is the ordinary genus of the curve $g(X)$. The *irregularity* of a n-fold X is the number $q(X) = h^{1,0}(X) = h^0(\Omega^1[X]) = h^1(\Omega^0[X]) = h^1(\mathcal{O}_X)$. The Plurigenera $P_n(X)$ of a variety X are the dimensions $h^0(nK_x)$, where K_X is the canonical divisor on X.

For any curve C on a surface S, we have $g(C) \leq \frac{C \cdot K_C + C \cdot C}{2} + 1$. Indeed, for a smooth curve C, by the Riemann-Roch formula applied to the canonical divisor K_C(Appendix 4.1) we have,

$$\begin{aligned} 1 - g(C) + degK_C &= \ell(K_C) - \ell(K_C - K_C) \\ &= g(C) - 1 \end{aligned}$$

that is

$$deg(K_C) = 2g(C) - 2.$$

Together with the adjunction formula

$$K_C = (K_S + C)|_C$$

(Appendix 4.2) we see that

$$g(C) = \frac{C \cdot K_S + C \cdot C}{2} + 1.$$

If C is non-singular then by the Riemann-Hurwitz formula (Appendix 4.3)

$$g(C) = \frac{C \cdot K_S + C \cdot C}{2} + 1 - ram_f$$

where the map f is a map given by a global rational function on S (i.e. if $f \in k(S)$ written locally as $f = g/h$, then $s \mapsto (g(s) : h(s))$ gives a map from S to $\mathbb{P}^1$).

We define the *virtual genus*, $\pi(C)$, to be the number $\frac{C \cdot K_S + C \cdot C}{2} + 1$, in what follows this will also be referred to as the adjunction formula. The relationship between the genus and the virtual genus is clear; if $\varphi : \tilde{C} \to C$ is the normalisation of the curve C then $\pi(C) = g(\tilde{C})$. In particular, since $g(C)$ is always non-negative, we have from the above that $\pi(C)$ is always non-negative. Furthermore, since a curve is rational if, and only if, it has genus zero, we see that a curve with $\pi(C) = 0$ is rational.

Finally, before moving on to look at rationality criterion, we present a criterion for determining when a curve is an exceptional curve of the first kind – and so may be blown down.

Corollary 2. *(Castelnuovo-Enriques criterion for blowing down)*
An irreducible curve C on a surface S may be blown down if, and only if,

$$C \cdot C < 0 \qquad \text{and} \qquad K_S \cdot C < 0.$$

Proof.
By the adjunction formula,

$$
\begin{aligned}
\pi(C) &= \frac{C\cdot C + K_S\cdot C}{2} + 1 \\
&\geq g(C) \\
&\geq 0.
\end{aligned}
$$

Whence $C\cdot C = K_S\cdot C = -1$, if $C\cdot C < 0$ and $K_S\cdot C < 0$. Thus $\pi(C) = 0$ and C is rational with self-intersection -1. That is to say, C is an exceptional curve of the first kind and so we may blow it down. □

2.2. Rationality criterion

Next, on towards the promised rationality criterion.

Lemma 3. *(Noether's Lemma) A surface is rational if, and only if, it contains an irreducible rational curve C with $h^0(C) - 1 = dim|C| \geq 1$.*

Proof.
The forward implication is clear; if $\varphi : S \dashrightarrow \mathbb{P}^2$ is a birational map, then we can pullback a general hyperplane H, that is take $C = \varphi^* H$.

Conversely, suppose that $C \subset S$ is an irreducible rational curve with $dim|C| \geq 1$. Choose a pencil $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ which contains C. We claim that on blowing up S sufficiently many times at the base points of the pencil $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ we obtain a surface $\tilde{S}$ on which the proper transforms $\{\tilde{C}_\lambda\}_{\lambda\in\mathbb{P}^1}$ of the curves C_λ form a pencil without base points. Clearly all the $\tilde{C}_\lambda$ will remain rational.

Indeed, suppose the linear system $|C|$ has a base point $p \in S$ of multiplicity r (i.e. p has multiplicity r on a generic $D \in |C|$). Let $\sigma : \tilde{S} \to S$ be the blow up of S at p, with $E = \sigma^{-1}(p)$ the exceptional divisor. Then

$$\tilde{D} = \sigma^{-1}(D) - rE$$

for $D \in |C|$.

$\tilde{C}$ has self-intersection

$$
\begin{aligned}
\tilde{C}\cdot\tilde{D} &= (\sigma^{-1}(D) - rE)(\sigma^{-1}(D) - rE) \\
&= \sigma^{-1}(D)\cdot\sigma^{-1}(D) - 2rE\cdot\sigma^{-1}(D) + r^2 E\cdot E \\
&= D\cdot D - r^2 \qquad (\text{since } E^2 = -1,\ \sigma^{-1}(D) \sim D \text{ and } D\cap E = \emptyset) \\
&< D\cdot D
\end{aligned}
$$

Now we construct a sequence of blowups $\sigma_i : S_i \to S_{i-1}$ and linear systems $|C_i|$ on S_i as follows: Let $\sigma_1 : S_1 \to S$ be the blowup of S at the base points of $|C|$ and $|C_1|$ the proper transform of $|C|$ in

S_1, $\sigma_2 : S_2 \to S_1$ be the blowup of S_1 at the base points of $|C_1|$ and $|C_2|$ the proper transform of $|C_1|$ in S_2, etc.

If every system were to have base points, then we would have

$$D \cdot D > D_1 \cdot D_1 > D_2 \cdot D_2 > \cdots$$

but the D_i are effective, so that $D_i \cdot D_i \geq 0$. Hence for some i the linear system $C_i|$ is base point free.

Therefore, we may assume from the start that S contains a base point free pencil $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ of rational curves, not all reducible.

We show now that we may find a surface birational to S with a pencil of irreducible disjoint rational curves. Such a surface, called a *geometrically ruled surface*, is rational and the proof will be complete. We shall use the Castelnuovo-Enriques criterion for blowing down to this end.

Any point of intersection of two distinct curves $\mathbb{C}_\lambda, C_{\lambda'} \in \{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a base point of $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$. Thus,

$$C_\lambda \cdot C_\lambda = C_\lambda \cdot C_{\lambda'} = 0.$$

Suppose $C_\mu \in \{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ is reducible, then we may write $C_\mu = \sum \alpha_\nu C_\nu$ with all $\alpha_\nu > 0$ and C_ν irreducible. Since each C_ν is disjoint from any $C_\lambda \sim C_\mu$ for $\lambda \neq \mu$,

$$0 = C_\mu \cdot C_\nu = \sum_{\nu'} \alpha_{\nu'} (C_{\nu'} \cdot C_\nu)$$

However, $C_{\nu'} \cdot C_\nu \geq 0$ for $\nu \neq \nu'$ and so $C_{\nu'} \cdot C_\nu > 0$ for some $\nu \neq \nu'$. It follows that $C_\nu \cdot C_\nu < 0$ for all ν.

The adjunction formula applied to the rational curve C_μ yields

$$\pi(C_\mu) = \frac{C_\mu \cdot C_\mu + K_S \cdot C_\mu}{2} + 1 = 0$$

Thus

$$C_\mu \cdot K_S = \sum \alpha_\nu C_\nu \cdot K_S = -2$$

and so

$$C_{\nu_0} \cdot K_S < 0$$

for some ν_0.

We have now both

$$C_{\nu_0} \cdot K_S < 0 \qquad \text{and} \qquad C_{\nu_0} \cdot C_{\nu_0} < 0$$

and so, by the Castelnuovo-Enriques blowing down criterion, we may blow C_{ν_0} down. Let $\varphi : S \to S'$ be the blowing down of C_{ν_0}.

Observe that every curve C_λ, other than C_μ, is disjoint from C_{ν_0}. We see then that the curves $\varphi(C_\lambda)$ form a base point free pencil of rational curves on S'.

Repeating the argument; if any curve $\varphi(C_\lambda)$ is reducible then we may blow down S' along it. Since we may blow down a surface only a finite number of times, a finite number of steps yields a surface $\hat{S}$ and a birational map $\psi : S \dashrightarrow \hat{S}$, such that the curves $\psi(C_\lambda)$ form a pencil of irreducible disjoint rational curves. Therefore we have shown that S is birational to a geometrically ruled surface and hence rational. □

From Noether's lemma follows Castelnuovo's criterion, which classifies all the rational surfaces as those sharing two simple numerical invariants. This strong statement is proved, rather surprisingly, from the repeated application of both the Riemann-Roch and adjunction formulae (with an appeal to Noether's formula (Appendix 4.4) in the final case). Immediately following the proof, we deduce from this, the Lüroth theorem for surfaces.

Theorem 4. *(Castelnuovo's Rationality Criterion, 1893)*
A surface S is rational over an algebraically closed field if, and only if, $q(S) = P_2(S) = 0$.

Proof.[3] ([Kodaira]) If S is rational then, as the plurigenera and irregularity of $\mathbb{P}^2$ are zero, we have $q(S) = P_2(S) = 0$.

Proving the other implication is slightly harder. Firstly, note that if S contains any (-1)-curves then we may blow them down to obtain a surface birational to S that has none. We assume then that S has no exceptional curves of the first kind.

We wish to apply Noether's lemma:

> A surface is rational if, and only if, it contains an irreducible rational curve C with $h^0(C) - 1 = dim|C| \geq 1$.

Thus, we wish to show that S contains a curve C with $g(C) = p_g(C) = \pi(C) = 0$ and $dim|C| \geq 1$.

For the proof we transpose the problem slightly; since $P_2(S) = 0$ we have $p_g(S) = 0$. Indeed, $p_g(S) = h^2(\mathcal{O}_S) = h^0(K_S)$, by Serre duality, and an element $\omega \in |K_S|$ yields an element $\omega^2 \in |2K_S|$, but $P_2(S) = h^0(2K_S) = 0$ and so $p_g(S) = 0$. Hence

$$\begin{aligned}\chi(\mathcal{O}_S) &= h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S) \\ &= 1 - q(S) + p_g(S) \\ &= 1.\end{aligned}$$

[3]Shown here is proof over $\mathbb{C}$, the theorem is true however, over any algebraically closed field. Over fields of arbitrary characteristic this was first proved by Zariski (see [Zariski58-1] and [Zariski58-2]), for alternative proofs see [Kurke] or [Lang]

By the Riemann-Roch theorem, applied to any curve C on S, we obtain

$$\chi(C) = \chi(\mathcal{O}_S) + \frac{C \cdot C - C \cdot K_S}{2}$$

that is

$$h^0(C) - h^1(C) = 1 + \frac{C \cdot C - C \cdot K_S}{2}$$

thus

$$h^0(C) \geq 1 + \frac{C \cdot C - C \cdot K_S}{2}$$

If C is a rational curve and $C \cdot C \geq 0$, then by the adjunction formula $\pi(C) = 1 + \frac{C \cdot C + C \cdot K_S}{2}$, $K \cdot C \leq -2$; hence $h^0(C) \geq 2$ and $dim|C|(= h^0(C) - 1) \geq 1$.

Therefore proof of the theorem is reduced to finding an irreducible curve C such that

$$\pi(C) = 0 \qquad \text{and} \qquad C \cdot C \geq 0$$

The proof splits into three cases; $K_S \cdot K_S < 0$, $K_S \cdot K_S = 0$ and $K_S \cdot K_S > 0$. We write below K for K_S.

Case: $K \cdot K = 0$.

Applying the Riemann-Roch formula to $-K$

$$\begin{aligned} \chi(-K) &= h^0(-K) - h^1(-K) + h^2(-K) \\ &= \chi(\mathcal{O}_S) + \frac{(-K)(-K-K)}{2} \\ &= 1 \end{aligned}$$

and so $h^0(-K) + h^2(-K) \geq 1$.
By Serre duality,

$$h^2(-K) = h^0(K - (-K)) = h^0(2K) = P_2(S)$$

which is zero by assumption. Thus $h^0(-K) \geq 1$ and it follows that there exists an effective divisor D linearly equivalent to $-K$, with D non-zero since K is non-trivial.

Let E be a very ample divisor on S.[4] We may assume that $h^0(E - D) \neq 0$. Since E is ample we have

$$E \cdot K = -E \cdot D < 0$$

hence $E \cdot (E + mK) = E \cdot E + mE \cdot K < 0$ for $m \gg 0$. Suppose that $E + mK$ were linearly equivalent to an effective divisor, then $E \cdot (E + mK) > 0$. This contradiction ensures that $h^0(E + mK) = 0$ for $m \gg 0$.

Choose now an n, such that

[4] E has the properties (i) $E^2 > 0$ (ii) $E \cdot C > 0$ for all curves C on S and (iii) the linear system $|E|$ has no base points, ie $|E|$ gives an embedding.

(i) $h^0(E+nK)>0$
(ii) $h^0(E+(n+1)K)=0$

and let $D' \in |E+nK|$ writing $D'=\sum \alpha_\nu C_\nu$. Then

$$K\cdot D' = K\cdot(E+nK) = K\cdot E + nK\cdot K = K\cdot E < 0$$

Hence $K\cdot C_{\nu_0}<0$ for some ν_0. Applying the Riemann-Roch formula to $-C_{\nu_0}$;

$$\chi(-C_{\nu_0}) = \chi(\mathcal{O}_S) + \frac{-C_{\nu_0}\cdot(-C_{\nu_0}-K)}{2} = 1+\frac{C_{\nu_0}\cdot C_{\nu_0}+C_{\nu_0}\cdot K}{2}$$

that is

$$h^0(-C_{\nu_0})-h^1(-C_{\nu_0})+h^2(-C_{\nu_0}) = \pi(C_{\nu_0})$$

thus

$$h^0(-C_{\nu_0})+h^2(-C_{\nu_0}) = h^0(-C_{\nu_0})+h^0(K+C_{\nu_0}) \geq \pi(C_{\nu_0})$$

by Serre duality. But clearly $h^0(-C_{\nu_0})=0$ and since $K+C_{\nu_0}<K+D'$, we have

$$h^0(K+C_{\nu_0}) \leq h^0(K+D') = h^0(K+(E+nK)) = h^0(E+(n+1)K)) = 0$$

by our choice of n. Therefore

$$h^0(-C_{\nu_0})+h^2(-C_{\nu_0}) = 0 \geq \pi(C_{\nu_0})$$

Thus

$$\pi(C_{\nu_0})=0$$

as $\pi(C_{\nu_0})\geq g(C_{\nu_0})\geq 0$. By the adjunction formula

$$0=\pi(C_{\nu_0}) = 1+\frac{C_{\nu_0}\cdot C_{\nu_0}+C_{\nu_0}\cdot K}{2}$$

we see that $C_{\nu_0}\cdot C_{\nu_0}\geq -1$. The self-intersection of C_{ν_0} cannot be -1 as there are no (-1)-curves on S by assumption. Hence $C_{\nu_0}\cdot C_{\nu_0}\geq 0$ and the theorem is proved in the case $K\cdot K=0$.

Case: $K\cdot K>0$**.**

Before proving the theorem for this case, we make a claim:
If E is any divisor on S then $h^0(E+nK)=0$ for $n\gg 0$. Indeed, choose n_0 large enough such that

$$K\cdot(E+n_0K) = K\cdot E+n_0K\cdot K<0$$

Suppose that $h^0(E+mK)\neq 0$ for some $m\geq n_0$. Let $D\in|E+mK|$, writing $D=\sum\alpha_\nu C_\nu$. Then

$$K\cdot D = K\cdot(E+mK) \leq K\cdot(E+n_0K)<0$$

thus $K\cdot C_{\nu_0}<0$ for some ν_0.

If $C_{\nu_0}\cdot C_{\nu_0}<0$ then, by the adjunction formula, we would have $K\cdot C_{\nu_0}=C_{\nu_0}\cdot C_{\nu_0}=-1$. However, we assumed that S has no exceptional curves of the first kind. Therefore $C_{\nu_0}\cdot C_{\nu_0}\geq 0$ and so $C_{\nu_0}\cdot D'\geq 0$ for any effective divisor D'. Since $K\cdot C_{\nu_0}<0$ we see that for $l\gg 0$

$$(E+lK)\cdot C_{\nu_0}<0 \qquad \text{would imply that} \qquad h^0(E+lK)=0.$$

This contradiction ensures the validity of our claim.

To start the proof of this case, let E be a very ample divisor with $h^0(E+K) \geq 2$ and choose an n such that

$$h^0(E+nK) \geq 2 \qquad \text{and} \qquad h^0(E+(n+1)K) \leq 1.$$

Let D be a generic element of $|E+nk|$. Then we claim that if we write $D = F + \sum \gamma_\nu C_\nu$, where F is the fixed component[5] of $|D|$, then $C_\nu \cdot C_\nu \geq 0$ for all ν. Indeed, consider the reduced linear system $|D'| = |D-F|$; $|D'|$ has only isolated base points. Blowing up at the base points yields a linear system $|\tilde{D'}|$, with elements $\tilde{D'_\lambda} = \sum \tilde{\gamma_\nu} \tilde{C_{\nu_\lambda}}$. Since any point of intersection between $\tilde{C_{\nu_\lambda}}$ and $\tilde{C_{\nu'_\lambda}}$ would be a singular point of D'. It is clear that for a generic (and hence smooth) λ the $\tilde{C_{\nu_\lambda}}$ are disjoint. Thus

$$\tilde{C_{\nu_\lambda}} \cdot \tilde{C_{\nu_\lambda}} = \tilde{C_{\nu_\lambda}} \cdot D' = 0$$

and so $C_\nu \cdot C_\nu \geq 0$ for all ν, as we claimed.

Since $h^0(-C_\nu) = 0$, by Riemann-Roch applied to $-C_\nu$, we have

$$h^0(K+C_\nu) \geq \frac{C_\nu \cdot C_\nu + C_\nu \cdot K}{2} + 1 = \pi(C_\nu)$$

but

$$h^0(K+C_\nu) \leq h^0(K+D) = h^0(K+(E+nK)) = h^0(E+(n+1)K) \leq 1$$

by assumption. Therefore, either $\pi(C_\nu) = 0$ or $\pi(C_\nu) = 1$.

If $\pi(C_\nu) = 0$, then we are done as $C_\nu \cdot C_\nu \geq 0$. Suppose then, that $\pi(C_\nu) = 1$. Then $h^0(K+C_\nu) = h^0(K+D) = 1$. Let $D' \in |K+C_\nu|$ writing $D' = \beta_\mu E_\mu$. D' is non-zero; $D' = 0$ implies $0 \sim K + C_\nu$, which in turn implies that $K \sim -C_\nu$ so that $K \cdot K = C_\nu \cdot C_\nu < 0$. Since $\pi(C_\nu) = 1$, we have via the adjunction formula, $K \cdot K = -C_\nu \cdot C_\nu$ hence $D' \cdot C_\nu = (K+C_\nu) \cdot C_\nu = 0$. The E_μ are irreducible so that $C_\nu \cdot C_\nu \geq 0$ implies that $C_\nu \cdot E_\mu \geq 0$ for all μ. Consequently on considering $D' \cdot C_\nu$ we see that for all μ, $E_\mu \cdot C_\nu = 0$. Since $K \cdot C_\nu < 0$ and $K \cdot K < 0$ we have $D' \cdot K = (K+C_\nu) \cdot K < 0$ and hence $E_{\mu_0} \cdot K < 0$ for some E_{μ_0}.

However,

$$0 > E_{\mu_0} \cdot K = E_{\mu_0} \cdot (K + C_\nu) = E_{\mu_0} D' = \sum_\mu \beta_\mu E_\mu \cdot E_{\mu_0} \geq E_{\mu_0} \cdot E_{\mu_0}.$$

Consequently, the adjunction formula applied to E_{μ_0} yields

$$E_{\mu_0} \cdot K = E_{\mu_0} \cdot E_{\mu_0} = -1$$

that is to say, E_{μ_0} is an exceptional curve of the first kind. Impossible!

Case: $K \cdot K < 0$.

[5] a fixed component of $|D|$ is a divisor F such that $D' - F \geq 0$ for all $D' \in |D|$, that is F belongs to the base locus of $|D|$

Applying the Riemann-Roch formula to $-K$,

$$h^0(-K)+h^2(-K) \geq \frac{K\cdot K+K\cdot K}{2}+1>1$$

but

$$h^2(-K)=h^0(K-(-K))=h^0(2K)=P_2(S)=0.$$

Thus $|-K|$ contains a pencil of curves. Let D be a generic element of $|-K|$. Then, by the above, we may write $D=F+\sum \alpha_\nu C_\nu$ where F is the fixed component of $|D|$ and $C_\nu \cdot C_\nu \geq 0$ for all ν.

If D is reducible, that is $D \neq C_1$, then we have

$$h^2(-C_1)=h^0(K+C_1)=h^0(C_1-D)=h^0(C_1)-F-\sum \alpha_\nu C_\nu)=h^0(-(\alpha_1-1)C_1-F-\sum_{\nu\neq 1}\alpha_\nu C_\nu)=0$$

and clearly $h^0(-C_1)=0$. Thus, by the Riemann-Roch formula for $-C_1$,

$$0 \geq \frac{C_1\cdot C_1+K\cdot C_1}{2}+1=\pi(C_1)$$

and so $\pi(C_1)=0$ and since $C_1\cdot C_1 \geq 0$ we are done.

Assume now that D is irreducible, that is $D=C_1$. Firstly, $D\cdot K=-D\cdot D$ since $D\sim -K$ and so $\pi(D)=1$.

If every very ample divisor on S were a multiple of K, it would follow that every line bundle on S is a multiple of K, that is to say,

$$H^2(S,\mathbb{Z})=H^{1,1}(S,\mathbb{Z})\cong\mathbb{Z}$$

with $c_1(K)$ as a generator. Then by Poincaré duality $K\cdot K=1$.

Using Noether's formula (Appendix 4.4) we see that this is a contradiction:

$$1=\chi(\mathcal{O}_S)=\frac{K\cdot K+\chi(S)}{12}=\frac{1+3}{12}$$

Thus, we may choose a very ample divisor E on S such that E is not a multiple of K and furthermore, such that $h^0(E+K)\geq 1$. Since $E\cdot K=-E\cdot D<0$[6] we observe that $E\cdot(E+nK)<0$ for $n\gg 0$ and so $h^0(E+nK)=0$ for $n\gg 0$.

Choose an integer n_0 such that

$$h^0(E+n_0K)\geq 1 \qquad \text{and} \qquad h^0(E+(n_0+1)K)=0.$$

Take now, a generic element $D'=\sum\beta_\nu B_\nu$ of $|E+n_0K|$. D' must be non-zero, for E is not a multiple of K. D id effective, so that

$$K\cdot B_\nu=-D\cdot B_\nu\leq 0$$

[6] L ample and D effective $\Rightarrow L\cdot D>0$

for all ν. Again, we observe that

$$h^0(K+B_\nu) \leq h^0(K+D') = 0 \qquad \text{and} \qquad h^0(-B_\nu) = 0.$$

Applying the Riemann-Roch formula to $-B_\nu$,

$$0 \geq \frac{B_\nu \cdot B_\nu + K \cdot B_\nu}{2} + 1 = \pi(B_\nu)$$

hence $\pi(B_\nu) = 0$.

$K \cdot B_\nu \leq 0$; if $K \cdot B_\nu < -1$, then $B_\nu \cdot B_\nu = 0$ and we're done. If $K \cdot B_\nu = -1$, then $B_\nu \cdot B_\nu = -1$, a contradiction. Consider then the case where $K \cdot B_\nu = 0$ and $B_\nu \cdot B_\nu = -2$.

Apply the Riemann-Roch formula to the divisor $D - B_\nu$,

$$h^0(D-B_\nu) = h^2(D-B_\nu) \geq \frac{2D \cdot D + B_\nu \cdot B_\nu}{2} + 1 = D \cdot D = K \cdot K > 0$$

and

$$h^2(D-B_\nu) = h^0(K-D+B_\nu) = h^0(2K+B_\nu) \leq h^0(2K+D) = h^0(2K+(E+n_0K)) = h^0(E+(n_0+2)K) = 0$$

hence

$$h^0(D-B_\nu) > 0.$$

Let $H \in |D - B_\nu|$ and write $H = \sum \gamma_\nu A_\nu$. H must be non-zero, for otherwise we would have $B_\nu \sim D \sim K$ and then $K \cdot K = B_\nu \cdot B_\nu = -2$.

Applying the Riemann-Roch formula to $-A_\nu$,

$$h^0(-A_\nu) + h^2(-A_\nu) \geq \frac{A_\nu \cdot A_\nu + K \cdot A_\nu}{2} + 1 = \pi(A_\nu)$$

but

$$h^2(-A_\nu) = h^0(K+A_\nu) \leq h^0(K+H) = h^0(-A_\nu) = 0$$

It follows that $\pi(A_\nu) = 0$. However,

$$H \cdot K = (-K - B_\nu) \cdot K = -K \cdot K < 0$$

so that $A_{\nu_0} \cdot K < 0$ for some ν_0. Therefore, either $A_\nu \cdot A_\nu = -1$ or $A_\nu \cdot A_\nu = 0$. A_ν cannot have self-intersection -1 by assumption, thus $A_\nu \cdot A_\nu = 0$ and this completes the proof. □

2.3. A Lüroth theorem for surfaces

Theorem 5. *(Lüroth Theorem for surfaces)*
If a surface S is unirational over an algebraically closed field k, such that the extension $k(x_1, x_2)$ over $k(S)$ is separable. Then it is, in fact, rational over k.

Proof.
This follows immediately from Castelnuovo's rationality criterion. Indeed, suppose that S is a unirational surface over an algebraically closed field k of characteristic 0 or that k is algebraically closed of characteristic p and the extension of $k(x_1,x_2)$ over k is separable. Either of these situations imply that the pullback of the dominant rational map $\varphi : \mathbb{P}^2 \dashrightarrow S$ will define an inclusion $\varphi^*\Omega^r[\mathbb{P}^2] \subseteq \Omega^r[S]$. More specifically, these conditions on k exclude the possibility of the existence of an inseparable map where each point is a ramification point (for example, $x \mapsto x^p$ over a field of characteristic p).

Thus, if $P_2(S)$ is non-zero, then the pullback of some non-zero $\omega \in \Omega^2[S]$ has to vanish on $\mathbb{P}^2$. Hence the Jacobian of φ must be zero everywhere – but this contradicts the fact that $\varphi : U \to V$ is surjective, by the Implicit Function theorem. Similarly, if $q(S) = h^1(\mathcal{O}_S) \neq 0$ then the pullback of a non-zero regular one-form will vanish on $\mathbb{P}^2$, so that again the Jacobian of φ will be identically zero. $\square$

2.4. Counterexamples over fields of characteristic $p > 0$

In [Zariski58-2] (see also [Katsura]), we see an example of a non-rational unirational surface over a field k of characteristic $p \geq 3$ which we shall present here to show the necessity of the requirement that k is perfect.[7]

Consider the affine surface S defined by the equation

$$f(x_0,x_1,x_2) = x_0^p + x_1^{p+1} + x_2^{p+1} - \frac{(x_1^2 + x_2^2)}{2} = 0$$

and the projective completion $\bar{S}$ of S. On S we have

$$x_0 = -x_1^{(p+1)/p} - x_2^{(p+1)/p} + \frac{(x_1^{2/p} + x_2^{2/p})}{2^{1/p}}$$

thus,

$$k(\mathbb{P}^2) = k(\mathbb{A}^2) = k(x_1^{1/p}, x_2^{1/p}) = k(x_0, x_1^{1/p}, x_2^{1/p}) \supseteq k(x_0,x_1,x_2) = k(\bar{S})$$

hence $\bar{S}$ is unirational.

However, observe that the regular two-form defined by

$$\omega = \frac{dx_0 \wedge dx_2}{\partial f / \partial x_1} = \frac{dx_0 \wedge dx_2}{x_1^p - x_1}$$

is non-zero on $\bar{S}$. It follows from Castelnuovo's rationality criterion (Theorem 4) that $\bar{S}$ is non-rational.

[7]Counterexamples can also be given in fields of characteristic 2 see, for example [Zariski58-2] or [Shioda].

2.5. **Counterexamples over algebraically non-closed fields**

We claim that the cubic surface S_3, given by

$$3x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

is unirational, but not rational over $\mathbb{Q}$.

In working through the details and justifying this claim, we use the following results of Segre, later reworked and improved, by Manin and Kollár (see [Kollár], [Manin], [Segre42], [Segre43], [Segre51] or [C-K-S]).

Theorem 6. *(Kollár, 2002)*[8]
A smooth cubic hypersurface of dimension at least two is unirational over k if and only if it admits a k-point.

Theorem 7. *(Segre, 1942* [Segre42]*(Restated, see* [C-K-S]*))*
No smooth cubic surface of Picard number one is rational over k.

Thus to proof of the claim is reduced to showing that S_3, (i) admits a k-point and, (ii) has Picard number one.

Recall that a k-point of a projective variety X is a point $x \in X$ such that all the coordinates of x are members of the field k. We see immediately that our surface S_3 has a $\mathbb{Q}$-point, namely the point $(-1:1:1:1)$. Thus, by Theorem 6, S_3 is unirational over $\mathbb{Q}$.

As is often the case proving the non-rationality of a variety is a difficult problem. To show that the Picard number[9] is one, we appeal to a further theorem of Segre see ([Segre51]).

Theorem 8. *(Segre, 1951) Let S_k be a smooth cubic surface in $\mathbb{P}^3$ and consider the action of the Galios group of $\frac{K}{k}$ on the 27 lines of S_K. Where K is a finite extension of k such that the 27 lines of $S_{\bar{k}}$ are defined over K. The following are equivalent.*

(i) *The Picard number $\rho_k(S)$ is one.*
(ii) *The sum of the lines in each Galios orbit is linearly equivalent to a multiple of the hyperplane class on S.*
(iii) *No Galios orbit consists of disjoint lines on S.*

The 27 lines on a cubic surface over an algebraically closed field are very well studied (see, for example, [G-H], [Shaf1] or [Reid]) and there is a wealth of literature on the subject. We recall here, that for a smooth cubic surface X defined over an algebraically closed field by

$$\alpha_0 x_0^3 + \alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_3 x_3^3 = 0$$

---

[8] Segre, [Segre51] in 1951, proved the restriction to the case where k is a perfect infinite field and the k-point is not an Eckardt point

[9] the Picard number of a variety is the rank of the Néron-Severi group or equivalently, if the variety is smooth, the rank of the Picard group

We may factor

$$(\alpha_0 x_0^3 + \alpha_1 x_1^3) + (\alpha_2 x_2^3 + \alpha_3 x_3^3)$$

into linear factors, distinct due to the non-singularity of X, as

$$l_1 l_2 l_3 + m_1 m_2 m_3$$

this yields nine lines defined by the nine pairs of planes that contain them

$$\{l_i = m_j = 0\}_{ij}$$

The other two pairings $(\alpha_0 x_0^3 + \alpha_2 x_2^3) + (\alpha_1 x_1^3 + \alpha_3 x_3^3)$ and $(\alpha_0 x_0^3 + \alpha_3 x_3^3) + (\alpha_1 x_1^3 + \alpha_2 x_2^3)$ produce the other eighteen lines on X.

In a similar fashion (following the example given in [C-K-S]), we factor

$$(x_1^3 + x_2^3) + (x_3^3 + 3x_0^3)$$

as

$$(x_1 + x_2)(x_1 + \omega x_2)(x_1 + \omega^2 x_2) + (x_3 + \beta x_0)(x_3 + \beta\omega x_0)(x_3 + \beta\omega^2 x_0)$$

where $\beta = \sqrt[3]{3}$ and $\omega = e^{\frac{2\pi i}{3}}$. This produces nine lines given, as before, by the pairs of planes containing them. We consider the Galois orbits of these nine lines.

For $i = 0,1,2$, the Galois group acts transitively on the $\beta\omega^i$. Thus, the lines given by

$$\{x_1 + x_2 = 0; x_3 + \beta\omega^i x_0 = 0\}$$

consist of one Galois orbit, comprising of three lines in the plane $\{x_1 + x_2 = 0\}$. In particular, none of these lines are disjoint.

Considering the orbit of a line defined by

$$\{x_1 + \omega x_2 = 0; x_3 + \beta\omega^i x_0 = 0\}$$

for some $i = 0,1,2$. We observe that the permutation

$$\beta \mapsto \beta\omega \mapsto \beta\omega^2 \mapsto \beta$$

fixes ω therefore the orbit of this line consists, in part, of all three of the lines defined by

$$\{x_1 + \omega x_2 = 0; x_3 + \beta\omega^i x_0 = 0\}$$

for $i = 0,1,2$, furthermore they all lie in the plane $\{x_1 + \omega x_2 = 0\}$ and hence cannot be disjoint.

On observing that the remaining three lines

$$\{x_1 + \omega^2 x_2 = 0; x_3 + \beta\omega^i x_0 = 0\}$$

for $i = 0,1,2$ can be obtained from these lines via the permutation interchanging $\beta\omega$ and $\beta\omega^2$ so that these three lines lie in the plane $\{x_1 + \omega^2 x_2 = 0\}$ and hence cannot be disjoint. We see that the orbit consists of six lines lying in two planes.

Consideration of the other two pairings of the equation defining our surface, we find two more orbits of three lines in the same plane and two orbits of six lines in two planes. None of the orbits consists of disjoint lines, hence the Picard number of S_3 is one.

What was special about our surface S_3? As can be seen from the above it is that 3 is not a perfect cube. Thus, with the above reasoning we may conclude that any cubic surface defined by

$$\alpha x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

where α is not a perfect cube is non-rational.

In fact, Segre (see [Segre51]) generalised this example to obtain the stronger result below.

Theorem 9. *(Segre, 1951) Any surface defined over $\mathbb{Q}$ by the following equation*

$$\alpha_0 x_0^3 + \alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_3 x_3^3 = 0$$

has Picard number one if and only if, for all permutations σ of four letters, the rational number

$$\frac{\alpha_{\sigma(0)}\alpha_{\sigma(1)}}{\alpha_{\sigma(2)}\alpha_{\sigma(3)}}$$

is not a perfect cube.

2.6. The Noether-Fano method

Before we go on to discuss higher dimensions, let's look in more detail at Theorem 7. The method of the proof has become known as the Noether-Fano method; it can be applied in higher dimensions and leads to the proof of the non-rationality of the smooth quartic three-folds.

Our first observation and the starting point of the proof; the Picard group of our cubic surface S_3 is generated by the class of a hyperplane section H, or, equivalently, by the anti-canonical class – since by the adjunction formula we have, $K_{S3} \sim -H$. Furthermore, the hyperplane section class H is not divisible. Indeed, if $H = mD$ for some integer m and divisor D, then $H^2 = m^2 D^2$. Since $H^2 = 3$, m must be one.

Suppose Theorem 7 is false. Then there exists a birational map $\varphi : S_3 \dashrightarrow \mathbb{P}^2$ defined over k. Associated to φ is a mobile linear system Γ, that is Γ is a linear system with no fixed curves. Since the Picard group of S_3 is generated by the hyperplane class H, Γ must be contained in the complete linear system $|dH|$ for some d. To prove Theorem 7 then, we prove the following.

Theorem 10. *If S_3 is a smooth cubic surface contained in $\mathbb{P}^3_k$, then there is no mobile linear system on S contained in $|dH|$.*

Although this new theorem doesn't have the appeal of Segre's theorem above, we have reduced the proof to that of one over an algebraically closed field. Indeed, we may assume that k is algebraically closed since a linear system is defined over a non-algebraically closed field even if its base points are not.

To begin the proof of Theorem 10, suppose there exists a mobile linear system $\Gamma \subseteq |dH|$ for some d. Then Γ defines a birational map $\varphi_\Gamma : S_3 \dashrightarrow \mathbb{P}^2_k$ and we may assume that k is algebraically closed. Let $P_1, \ldots, P_r$ be the base points of Γ with multiplicities $m_1, \ldots, m_r$ including the possible infinitely near base points. We claim that some base point must have multiplicity greater than d.

Before proving this claim, a digression. Suppose that Λ is a mobile linear system on a surface S. If Λ is base point free, then Λ defines a regular map ψ_Λ and the self-intersection of Λ, is $\Lambda^2 = deg(\psi_\Lambda) deg(\psi_\Lambda(S))$. Suppose that Λ has a base point P of multiplicity m. With ease we may check that $\pi^* C = C' + mE$ where C is any curve on S, $S' \to^\pi S$ is the blow up of S at P, C' is the birational transform of C and E is the exceptional divisor. Thus,

$$\Lambda' = \pi^* \Lambda - mE \qquad \text{and} \qquad K_{S'} = \pi^* K_S + E$$

and so

$$\Lambda'^2 = \Lambda^2 - m^2 \qquad \text{and} \qquad \Lambda' \cdot K_{S'} = \Lambda \cdot K_S + m.$$

In our case, $P_1, \ldots, P_r$ are base points with multiplicities $m_1, \ldots, m_r$. On iterating the process above we have

$$\Gamma'^2 = \Gamma^2 - \sum m_i^2 \qquad \text{and} \qquad \Gamma' \cdot K_{S'} = \Gamma \cdot K_S + \sum m_i$$

where $S' \to S$ is the blow up of S at the all base points of Γ. Moreover, since Γ' is base point free we see that $\Gamma'^2 = 1$ and that $\Gamma' \cdot K_{S'} = \varphi^*_{\Gamma'} H \cdot (\varphi^*_{\Gamma'} K_S + E_{\Gamma'}) = H \cdot K_S = -3$. Hence

$$1 = \Gamma^2 - \sum m_i^2 \qquad \text{and} \qquad -3 = \Gamma \cdot K_S + \sum m_i.$$

From this we see that

$$\sum m_i = 3d - 3 \qquad \text{and} \qquad \sum m_i^2 = 3d^2 - 1.$$

Now suppose that for all m_i, $m_i \leq d$. Then

$$3d^2 - 1 = \sum m_i^2 \leq d \sum m_i = d(3d-3) = 3d^2 - 3d < 3d^2 - 1$$

a contradiction! So our claim holds.

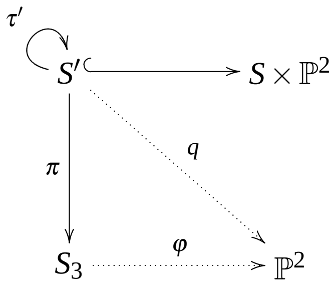
Next, let P be a base point of Γ of multiplicity greater than d. Without loss of generality, we may assume that P is a point on S (and not an infinitely near base point) since the multiplicity of a base point is greater than or equal to the multiplicity of a base point infinitely near. We also deduce that P cannot lie on any line in S. Indeed, $\Gamma \subset |dH|$ and so $L \cdot C \leq d$ for all lines L on S and all $C \in \Gamma$. $L \cdot C = \sum_{Q \in C \cap L} (multi_Q C)$ so no point $Q \in C$ can have multiplicity greater than d.

For the completion of the proof we proceed by induction; we find a birational automorphism of S (in fact this turns out to be a birational involution) that takes Γ to a linear system contained in the complete linear system $|d'H|$ with $d' < d$ – contradicting the minimality of d.

We are looking now for an birational automorphism of our surface S_3. The construction has a simple analogue for a cubic plane curve which we shall examine first. Let C be a plane cubic curve and fix a point $P \in C$. For all points $Q \in C$ send Q to the unique point of intersection Q' between

C and the line through P and Q. To extend this to an involution on C we send the point P to the point of intersection between the tangent line of C at P and the curve C. This is now a well-defined involution on C.

By analogy, let us try to construct an involution, τ, on S_3 in this way. First fix a point $P \in S_3$ and send any point $Q \in S_3$ to the unique third point of intersection between S_3 and the line through Q and P. This is well defined if we choose our point P such that there is no line on S_3 through P, but we confirmed above that if P is a base point of Γ of multiplicity greater than d then this is true. To extend this in the same way as before we run into a small problem. Namely, there are many lines tangent to S_3 at P – in fact, there is a whole tangent plane at P. The resolution of this problem is simple – we blow up S_3 at the point P. Let $S' \to^{\pi} S_3$ be the blow up of S_3 at the point P. Now, downstairs τ is a well-defined involution outside of the point P and the curve $D = S_3 \cap T_P S_3$. Upstairs, on S', we may take any point on the proper transform of D, D', and map it to a unique point on the exceptional divisor E of π. Whence, we have a well-defined involution τ' on S'. We have the picture:



where q is the projection map onto the second factor, $\mathbb{P}^2$, of $S \times \mathbb{P}^2$.

The map $q : S' \dashrightarrow \mathbb{P}^2$ can be described as follows. For $Q \in S \backslash E$ we may think that $Q \in S_3$, as the S and S' are isomorphic outside of the exceptional divisor, then $q(Q)$ is the line L in $\mathbb{P}^2$ through P and Q. For $Q \in E$, we may think of Q as a direction through P, and $q(Q)$ is the line L through P in the direction Q.

The fibre of q over a point $M \in \mathbb{P}^2$ consists of the two points Q_1 and Q_2 that together with P make up the intersection of $S \cap L$. The fibre is ramified when the two points Q_1 and Q_2 coalesce, i.e. $Q_1 = Q_2$. Let's find this ramification locus. We are free to choose coordinates, we choose them such that the P is the point $(0:0:0:1)$. Locally, S_3 is given by an equation of the form

$$f_1(x,y,z) + f_2(x,y,z) + f_3(x,y,z) = 0$$

where x,y,z are locally coordinates and the f_i are homogeneous of degree i. We may describe any line L through P by the parametric equations $(\alpha t, \beta t, \gamma t)$, which corresponds to the point $(\alpha : \beta : \gamma) \in \mathbb{P}^2$. The intersection $L \cap S_3$ is given by the solutions to

$$t f_1(x,y,z) + t^2 f_2(x,y,z) + t^3 f_3(x,y,z) = 0.$$

$t=0$ corresponds to the point P and Q_1, Q_2 may be found using the quadratic formula. The ramification locus is therefore given by the discriminant

$$f_2^2 - 4 f_1 f_3 = 0.$$

Note also that this is smooth as both S_3 and $\mathbb{P}^2$ are.

Before we complete the proof, let's make a couple of observations about τ'. Firstly, τ' is the unique non-trivial Galois automorphism of the $2:1$ cover S' of $\mathbb{P}^2$. Furthermore, τ' interchanges E the exceptional divisor and D', the proper transform of D. Also we can see that $|\pi^* H - E| = |q^*(L)|$ where L is any line in $\mathbb{P}^2$ and H is a hyperplane section of S_3.

To recap, we have a base point P of a mobile linear system $\Gamma \subset |dH|$ with multiplicity $m > d$. Let $\Gamma' = \pi^*\Gamma - mE$ be the proper transform of Γ under π. We are looking for a contradiction and we hope to find $d' < d$ such that $\Gamma \subset |d'H|$, which by induction will prove Theorem 10. Since $\Gamma \subset |dH|$ we have the following,

$$\Gamma' + (m-d)E = \pi^*\Gamma - dE \subseteq |\pi^*(dH) - dE)| = |d(\pi^*H - E)| = |q^*(dL)|.$$

The involution τ' preserves the pullback of any linear system from $\mathbb{P}^2$. Thus, on application of the involution τ' the linear system $\Gamma' + (m-d)E$ is mapped to another linear system inside $|q^*(dL)|$. We see that upstairs

$$\tau'(\Gamma' + (m-d)E) = \tau'(\Gamma') + (m-d)D' \subseteq |q^*(dL)| = |d(\pi^*H - E)| \subseteq |\pi^*(dH)|.$$

Downstairs on S_3 we have,

$$\tau(\Gamma) + (m-d)D \subseteq |dH|.$$

$D = S_3 \cap T_P S_3$ is a hyperplane section of S_3 hence,

$$\tau(\Gamma) \subseteq |(d-(m-d))H|.$$

However, $m > d$ and so we have

$$\tau(\Gamma) \subseteq |d'H|$$

with $d' < d$. This completes the proof of Theorem 10 and hence of Theorem 7.

Manin proved in 1966, arguing in much the same way as above (see [C-K-S] 2.1), the following theorem. This was the starting point for his joint work with Iskovskikh on the non-rationality of the smooth quartic three-fold that we will look at below before going on to examine the Lüroth problem in higher dimensions.

Theorem 11. *(Manin, 1966)*
Two smooth cubic surfaces defined over a perfect field of Picard number one are birational to one another if, and only if, they are projectively equivalent.

3. The Lüroth Problem in Higher Dimensions

Many years after Castelnuovo's rationality criterion, the Lüroth problem in dimensions greater than two remained an open problem. In 1971 three independent papers were published giving counterexamples to the Lüroth problem - firstly [I-M] followed by [C-G] with [A-M] in November of the same year. Indeed, over any field it is not true, in general, that a unirational variety is necessarily rational.

To summarise the different approaches given in [I-M], [C-G] and [A-M]; Artin and Mumford in [A-M], following a suggestion of Ramanujam, exploited the fact that the torsion subgroup of the third integral cohomology group of a smooth complex variety is a birational invariant and in particular is zero if the variety is rational. Artin and Mumford explicitly constructed examples of unirational varieties, in all dimensions over fields of any characteristic ($\neq 2$), with non-zero torsion.

Over ten years earlier, in 1959, Serre had shown that unirational and rational three-folds share almost all cohomological properties – but a few differences slipped through; the torsion subgroup being one. Another was the 'intermediate Jacobian' an Abelian variety obtained from the Hodge decomposition of the third integral cohomology group. The intermediate Jacobian plays a similar role to the Jacobian variety one uses in the study of divisors on a curve. Clemens and Griffiths in [C-G], showed that a rational three-fold satisfies a certain relationship on the intermediate Jacobian, they then go on to show that no smooth cubic hypersurface satisfies this relationship – hence no smooth cubic hypersurface in $\mathbb{P}^4$ is rational. In an appendix they remind the reader of the unirationality of cubic three-folds, giving a short construction.

In 1960, Segre showed that the smooth quartic three-fold X_4 defined by

$$x_0^4 + x_0 x_4^3 + x_1^4 - 6x_1^2 x_2^2 + x_2^4 + x_3^4 + x_3^3 x_4 = 0$$

is unirational, where x_0, x_1, x_2, x_3, x_4 are homogeneous coordinates on $\mathbb{P}^4$.

Let's take a brief look at constructing such unirational three-folds (see [Marchisio]; take a rational surface $S \subset X_4$ and fix both a point $p \in S$ and a hyperplane $H \subseteq \mathbb{P}^4$. Take the tangent cone $C_p(X_4)$ to X_4 at p and let Q_p be the intersection of $C_p(X_4)$ with the hyperplane H, this is a one dimensional conic. Consider the map $\varphi : S \times \{Q_p\}_{p \in S} \dashrightarrow S$ the image of φ is of dimension three, that is, φ is dominant. Now, for each $p \in X_4$, Q_p is a conic; therefore, φ is a conic bundle over S. If we can show that φ has a rational section, i.e. a rational map from an open subset of S to $S \times \{Q_p\}_{p \in S}$, then X_4 will be unirational.

It is an open problem whether all conic bundles over a rational surface are unirational. However, we can choose our surface so as φ does indeed admit a rational section.

The trick here is to choose our surface S such that it has separable asymptotes, that is S consists of two irreducible components over an algebraic extension of k. Then the conic bundle will admit a rational section. Indeed, if we take our surface to be a monoidal quartic surface, that is a surface with a unique singularity, then there exist simple checks for separable asymptotes. Observe that the three-fold given by Segre above contains the monoidal quartic surface defined by

$$x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0$$

with a unique triple point at $(0:0:0:1)$.

Marchisio (see [Marchisio]) showed, using these methods, a 54-dimensional family of unirational quartic hypersurfaces in $\mathbb{P}^4$.

Let's look finally at the first construction of non-rational unirational three-folds to appear in 1971; that of Iskovskikh and Manin in [I-M]. Following suggestions of Fano, they proved the following theorem which immediately implies the non-rationality of the smooth quartic three-folds.

Theorem 12. *(Iskovskikh and Manin, 1971)*
Let X_4 be a smooth quartic three-fold in $\mathbb{P}^4$. Then the group $Bir(X_4)$ of birational automorphisms coincides with the group $Aut(X_4)$ of biregular automorphisms.

The non-rationality follows from finiteness of the group of birational automorphisms $Bir(X_4)$ of X_4. Indeed, the linear system $|\mathcal{O}_{\mathbb{P}^4}(1)|_{X_4}|$ generated by the hyperplane sections of X_4 is invariant under the action of the group $Aut(X_4)$, since the divisor $-K_X$ belongs to $|\mathcal{O}_{\mathbb{P}^4}(1)|_{X_4}|$, by the adjunction formula. Therefore the group of automorphisms of X_4 consists of projective automorphisms and is thus finite (see [M-M]).

In fact they showed, using the Noether-Fano method (2.6), that there are no birational maps that are not isomorphisms between X_4 and a wide range of three-folds; for example, $\mathbb{P}^3$, any cubic in $\mathbb{P}^4$ and any three-fold fibred into rational surfaces. In time more examples of three-folds surfaces with this property emerged and they became known as *birationally rigid varieties* (see [Cheltsov]).

To prove that X_4 is non-rational then; if we suppose that there exists a birational map between X_4 and $\mathbb{P}^3$, then by the work of Iskovskikh and Manin in [I-M] it has to an isomorphism – this contradiction ensures the non-rationality of the smooth quartic hypersurfaces in $\mathbb{P}^4$ (a good summary of [I-M] can be found in [Cheltsov]).

Of note is also Saltman's approach which is outlined in [Shaf90]. He showed that Lüroth's question has a negative solution in the following situation. Suppose that G is a finite group of linear transformations of a vector space V over an algebraically closed field k. Then $k(V)$ is the field of rational functions of the coordinates of V and $k(V)^G$ is the field of invariants of G. He shows in [Saltman] (later simplified see [Bogomolov]) that although $k(V)^G$ is necessarily a subfield of $k(V)$, it is not necessarily isomorphic to a rational function field. The ideas in the proof are simpler than the ones given in the three papers of 1971. Moreover, the techniques used were available twenty years before 1971.

4. Appendix

We state here some well known results and sources for the diligent reader.

4.1. The Riemann-Roch theorem

Riemann-Roch for a divisor D on a smooth curve C;

$$h^0(D) - h^0(K_C - D) = 1 - g(C) + deg(D)$$

Riemann-Roch for a divisor D on a smooth surface S;

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{D \cdot D + K_S \cdot D}{2}$$

that is

$$h^0(D) - h^1(D) + h^2(D) = \chi(\mathcal{O}_S) + \frac{D \cdot D + K_S \cdot D}{2}.$$

Details can be found in [G-H], [Harts] and [Shaf1]/[Shaf2].

4.2. The adjunction formula

Relating the canonical divisors of smooth varieties $Y \subseteq X$;

$$K_Y = (K_X + Y)|_Y$$

or also, following our discussion on the virtual genus in Section 2;

$$\pi(C) = \frac{K_S \cdot C + C \cdot C}{2} + 1$$

where C is an irreducible curve on a surface S.

For details; [G-H], [Harts] or [Shaf1] and [Shaf2].

4.3. The Riemann-Hurwitz formula

When $f : X \dashrightarrow Y$ is a rational map with $f(X)$ dense in Y the Riemann-Hurwitz formula relates the Euler characteristic of X to that of Y, taking into account possible ramifications of the map.

$$\chi(X) = N\chi(Y) - \sum_{x \in X} (\nu(x) - 1)$$

where $\nu(x)$ is the ramification index of f at $x \in X$, as a shorthand we shall write the formula as

$$\chi(X) = N\chi(Y) - ram_f$$

For details see either [G-H] 2.1 or [Shaf2] VII.3.1.

4.4. **Noether's formula**

For a surface S we have Noether's formula for expressing the holomorphic Euler characteristic of S in terms of the Chern classes of S;

$$\chi(S) = \frac{c_1^2(S) + c_2(S)}{12}.$$

See [G-H] 4.6 for details.

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ANDREW WILSON

www.explicitlyambiguous.co.uk
in.one.ear.and.out.the.other@gmail.com

UNIVERSITY OF EDINBURGH SCHOOL OF MATHEMATICS, ROOM 5316, JCMB, KING'S BUILDINGS, EDINBURGH, EH9 3JZ