Smooth Exceptional del Pezzo Surfaces

Andrew Wilson

March 15, 2010

For a Fano variety $V$ with at most Kawamata log terminal (klt) singularities and a finite group $G$ acting bi-regularly on $V$, we say that $V$ is $G$-exceptional (resp., $G$-weakly-exceptional) if the log pair $(V, \Delta)$ is klt (resp., log canonical) for all $G$-invariant effective $\mathbb{Q}$-divisors $\Delta$ numerically equivalent to the anti-canonical divisor of $V$. Such $G$-exceptional klt Fano varieties $V$ are conjectured to lie in finitely many families by Shokurov ([Sho00, Pro01]). The only cases for which the conjecture is known to hold true are when the dimension of $V$ is one, two, or $V$ is isomorphic to $n$-dimensional projective space for some $n$. For the latter, it can be shown that $G$ must be primitive — which implies, in particular, that there exist only finitely many such $G$ (up to conjugation) by a theorem of Jordan ([Pro00]).

Smooth $G$-weakly-exceptional Fano varieties play an important role in non-rationality problems in birational geometry. From the work of Demailly (see [CS08, Appendix A]) it follows that Tian’s $\alpha_G$-invariant for such varieties is no smaller than one, and by a theorem of Tian such varieties admit $G$-invariant Kähler-Einstein metrics. Moreover, for a smooth $G$-exceptional Fano variety and given any $G$-invariant Kähler form in the first Chern class, the Kähler-Ricci iteration converges exponentially fast to the Kähler form associated to a Kähler-Einstein metric in the $C^\infty(\mathcal{V})$-topology. The term exceptional is inherited from singularity theory, to which this study enjoys strong links.

We classify two-dimensional smooth $G$-exceptional Fano varieties (del Pezzo surfaces) and provide a partial list of all $G$-exceptional and $G$-weakly-exceptional pairs $(S, G)$, where $S$ is a smooth del Pezzo surface and $G$ is a finite group of automorphisms of $S$. Our classification confirms many conjectures on two-dimensional smooth exceptional Fano varieties.

References


