1S Calculus

Week 18

University of Glasgow

Monday, 14^{th} January 2013

1.6 Change of variables

Faced with the integral, $\int f(x)\,dx$, one approach is to introduce a new variable $u = g(x)$ and write the integral with respect to this new variable. This is called a *substitution*.

Theorem

Suppose $F(x)$ is an antiderivative of $f(x)$, then

$$F\big(g(x)\big) \text{ is an antiderivative of } f\big(g(x)\big)\cdot g'(x)$$

and

$$\int f\big(g(x)\big)\cdot g'(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) = F\big(g(x)\big).$$

Example

By making the substitution $u = \sin x$, find $\displaystyle\int \sin^4 x \cos x\,dx$.

Example (Use a substitution to evaluate the following integrals)

i) Find $\displaystyle\int \frac{(\log x)^3}{x}\,dx$ using $u = \log x$.

ii) Find $\displaystyle\int (x^3+4)^{5/2}x^2\,dx$ using $u = x^3+4$.

A surplus factor usually stands out in an integrand if change of variables (substitution) is a good method to use. This surplus factor corresponds to $\dfrac{du}{dx}$. For example,

$$\int \cdots \cos x\,dx \to \frac{du}{dx} = \cos x \text{ so try } u = \sin x$$

$$\int \cdots \sec^2 x\,dx \to \frac{du}{dx} = \sec^2 x \text{ so try } u = \tan x$$

$$\int \cdots x^2\,dx \to \frac{du}{dx} = x^2 \text{ so try } u = x^3 \text{ or } (x^3 + \text{constant}).$$

1.6 Change of variables

Example (Use a substitution to evaluate the following integrals)

i) $\displaystyle\int (\cos x)^{3/2} \sin x \, dx.$

ii) $\displaystyle\int x\sqrt{x^2+1}\, dx.$

iii) $\displaystyle\int x^2 e^{-x^3}\, dx.$

iv) $\displaystyle\int \frac{e^x}{e^x+1}\, dx.$

v) $\displaystyle\int \frac{x}{x^4+1}\, dx.$

vi) $\displaystyle\int \sec^4 x \, dx.$

Exercise

Verify SI 12 using substitution,

$$\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c.$$

(more on this in Section 1.14)

1.7 Integrals of the form $\int \sin^m x \cos^n x \, dx$; $m, n \in \mathbb{Z}^+$

Strategy (for $\int \sin^m x \cos^n x \, dx$; $m, n \in \mathbb{Z}^+$)

m	n	*use*		
even	*odd*	$u = \sin x$ $(du = \cos x \, dx)$	&	$\cos^2 x = 1 - \sin^2 x$
odd	*even*	$u = \cos x$ $(du = -\sin x \, dx)$	&	$\sin^2 x = 1 - \cos^2 x$
odd	*odd*	*either* $u = \sin x$ *or* $u = \cos x$	&	$\sin^2 x + \cos^2 x = 1$
even	*even*	*use trigonometric identities*		*(example below)*

Example (Integrate the following trigonometric functions)

i) $\int \sin^2 x \cos^3 x \, dx$.

ii) $\int \sin^3 x \, dx$.

iii) $\int \sin^9 x \cos^3 x \, dx$.

iv) $\int \sin^2 x \cos^2 x \, dx$.