1S Calculus

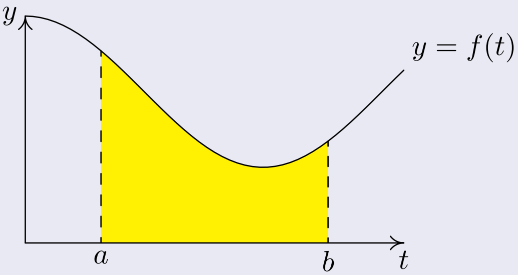
Proof of the Fundamental Theorem of Calculus

University of Glasgow

January 2013

Theorem (The Fundamental Theorem of Integral Calculus)

For a function f which is continuous on the interval $[a, b]$, the area of the region R indicated in the sketch below (the area between the graph of f, the t-atis and the lines $t = a$, $t = b$) is $\int_a^b f(t)\,dt$.



Area under the curve $y = f(t)$ between $t = a$ and $t = b$.

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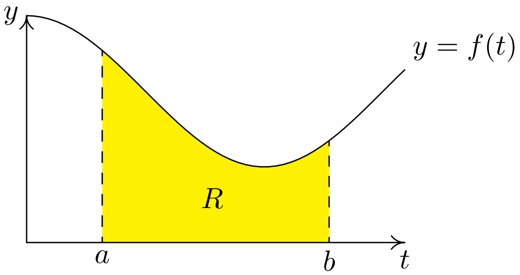

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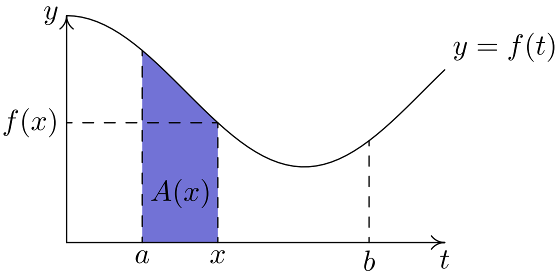

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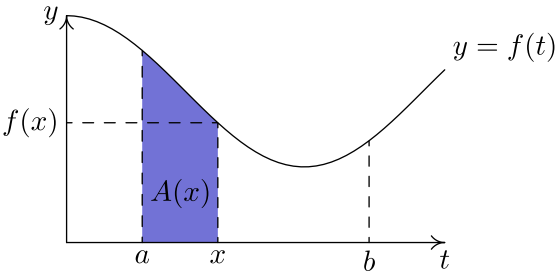

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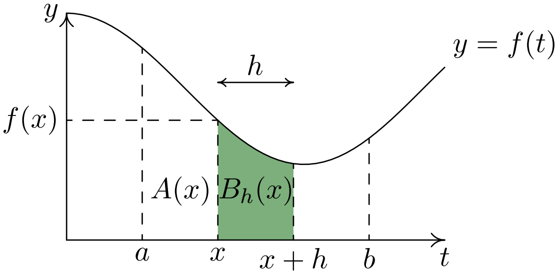

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- Consider the area $A(x)$, we'll show that $\frac{d}{dx}A(x) = f(x)$.
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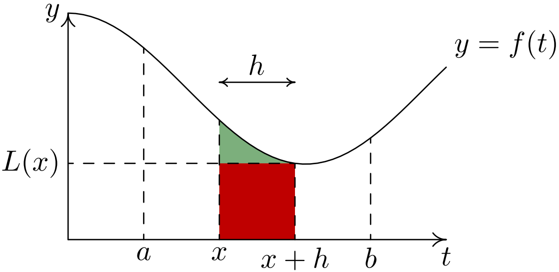

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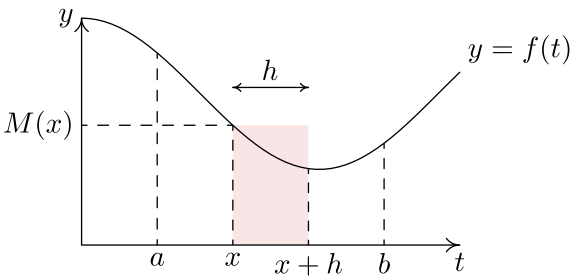

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- $$B_h(x) \leqslant h \cdot M(x)$$

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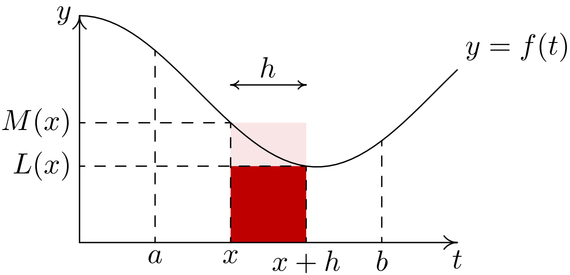

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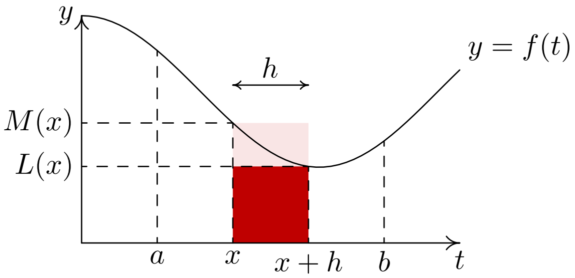

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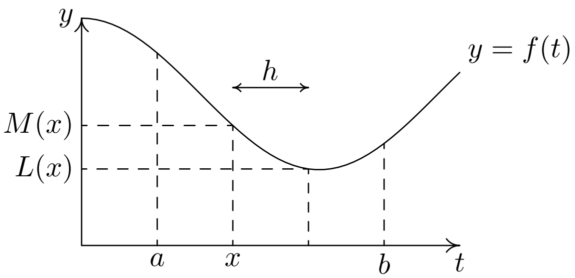

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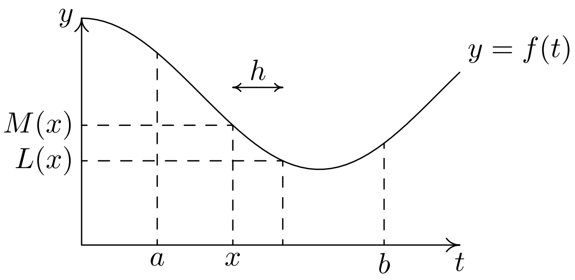

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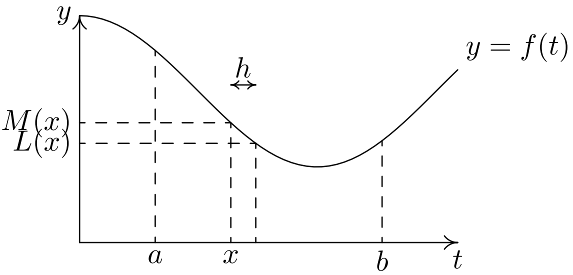

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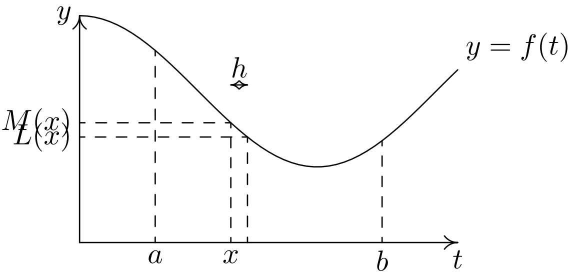

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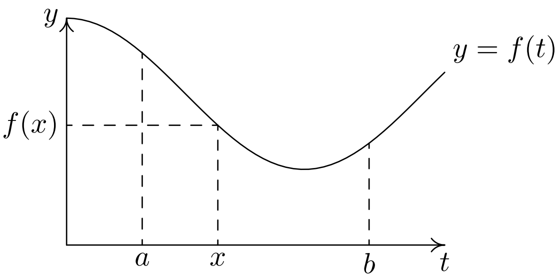

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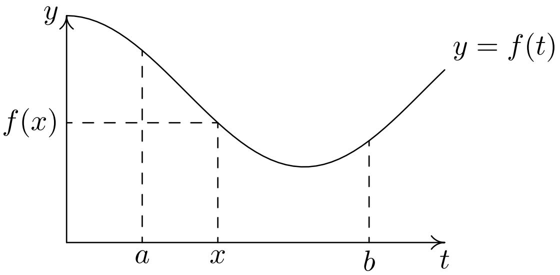
- As $h \longrightarrow 0$, $\quad L(x), M(x) \longrightarrow f(x)$.



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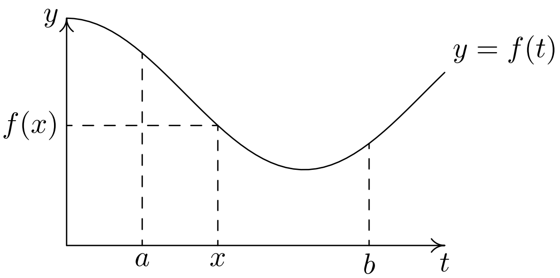

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- Thus, $\frac{B_h(x)}{h} \longrightarrow f(x)$ as $h \longrightarrow 0 \quad \Rightarrow \quad \frac{d}{dx}A(x) = f(x)$.