1S Calculus

Sections 1.16 – 1.17

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1.16 Trigonometric substitutions

Technique (Trigonometric substitutions)

integrand contains a factor	*try subsitution*
$(a^2 - x^2)^n$	$x = a \sin\theta$
$(a^2 + x^2)^n$	$x = a \tan\theta$

($a > 0$ constant)

Example (Use trig. substitution to calculate)

i) $\displaystyle\int \sqrt{4 - x^2}\, dx,$

ii) $\displaystyle\int_0^3 \frac{dx}{(x^2+9)^2}.$

Example

Use the change of variables $x = \sqrt{k}\tan\theta$ to establish the standard integral

$$\int \frac{dx}{\sqrt{x^2+k}} = \log|x + \sqrt{x^2+k}| + c,$$

where $k > 0$ is a constant.

Note that together with the standard integral

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

we can perform any integral of the form $\displaystyle\int \frac{1}{\sqrt{\text{quadratic}}}\,dx$.

Example (Calculate the integral)

$$\int \frac{dx}{x+\sqrt{x}+1}.$$

1.17 Symmetry and definite integrals

Definition (Odd and even functions)

A function $f(x)$ is
$$\begin{array}{lrcl} \text{even if} & f(-x) & = & f(x) \quad \forall x \\ \text{odd if} & f(-x) & = & -f(x) \quad \forall x. \end{array}$$

Lemma

Show that

i) *for an even function* f, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$,

ii) *for an odd function* g, $\displaystyle\int_{-a}^{a} g(x)\,dx = 0$.

Example

$$\int_{-2}^{2} 5x^7 + 6x^2 + 7x^3 + 9x^5\,dx$$

1.17 Symmetry and definite integrals

Technique (Integrals involving products of powers of $\sin x$ and $\cos x$)

Use the symmetry of the sine and cosine functions to simplify definite integrals involving their powers.

Example (Show that)

i) $\displaystyle\int_{\pi/2}^{\pi} \cos x\, dx = -\int_0^{\pi/2} \cos x\, dx$ ii) $\displaystyle\int_0^{\pi/2} \sin x\, dx = \int_0^{\pi/2} \cos x\, dx$

Example (Evaluate the following definite integrals)

i) $\displaystyle\int_0^{2\pi} \sin^2 x \cos x\, dx$

ii) $\displaystyle\int_0^{2\pi} \sin x \cos^3 x\, dx$

iii) $\displaystyle\int_0^{2\pi} \sin^2 x \cos^2 x\, dx$

iv) $\displaystyle\int_0^{\pi} \sin x \cos^4 x\, dx$