1S Calculus

Chapter 3 – Maclaurin series

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Definition

A **power series** in x is $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ for some constants a_n.

Definition

If the function $f(x)$ can be represented by a power series for some values of x then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

and we call the power series the **Maclaurin series** for $f(x)$. The values of x for which the infinite sum exists and is equal to $f(x)$ belong to the **range of validity** of the Maclaurin series.

Definition (Range of validity)

Consider

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N}_{\text{truncated power series}} + \underbrace{R_N(x)}_{\text{remainder}} .$$

Letting $N \to \infty$, we get $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for values of x for which $R_N(x) \to 0$ as $N \to \infty$ (this determines the range of validity).

Example

The range of validity for

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

is $(-1, 1]$.

3.1.1 Calculation of Maclaurin series

Technique (Method to calculate Maclaurin series coefficients a_i)

Suppose the Maclaurin series of a function $f(x)$ exists. Then we have, for x in the range of validity, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, where the coefficients are given by

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Example (Some standard Maclaurin series)

i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ valid for $x \in \mathbb{R}$.

ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ valid for $x \in \mathbb{R}$.

iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ valid for $x \in \mathbb{R}$.

iv) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ valid for $x \in (-1, 1]$.

3.1.1 Calculation of Maclaurin series

Example (Find the Maclaurin series for the following functions)

i) $f(x) = (1+x)^\alpha$

ii) $f(x) = (1-x)^{1/2}$.

iii) $f(x) = \dfrac{1}{2+x}$.

Technique

Consider the Maclaurin series for $f(x)$ and $g(x)$ and let I be the intersection of the range of validity for both series. Then the Maclaurin series for $f(x) + g(x)$, $f(x)g(x)$ and $f(g(x))$ can be found using the Maclaurin series for $f(x)$ and $g(x)$.

Example (Multiplication and composition of Maclaurin series)

i) Find the Maclaurin series for $\dfrac{e^{2x}}{1-x}$ as far as the term x^3.

ii) Find the Maclaurin series for $\log(\cos x)$ up to the term x^6.

3.2 Some applications of Maclaurin series

Idea

Suppose

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n}_{\textit{Truncated Maclaurin series}} + R_n(x)$$

When $|x|$ is small the first few terms in a Maclaurin series give a good approximation for $f(x)$ (the remainder is the error).

Example (Some applications)

i) Show that $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$, $|x| < 1$.
Take $x = \frac{1}{3}$ to find an approximation of $\log 2$.

ii) Fix x, then $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$. In particular,
$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

Example (Evaluate the following limits)

i) Find $\lim\limits_{x\to 0} \dfrac{e^{3x} - e^{x}}{x}$.

ii) Find $\lim\limits_{x\to 0} \dfrac{x\cos 2x - \sin x}{x^3}$.

Example (Taylor series)

The Maclaurin series of $f(x)$ is an expansion of the function about $x = 0$. In general we require $f(x)$ near $x = a$, say. An expansion of $f(x)$ about $x = a$ is called a *Taylor series*.

3.2 Some applications of Maclaurin series

Example (Hyperbolic cosine and sine)

Consider the Maclaurin series for sine and cosine

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots; \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

What functions have the following Maclaurin series?

$$S(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots; \qquad C(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

Definition (Hyperbolic sine and hyperbolic cosine)

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \qquad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

using the Maclaurin series it is easy to deduce the relations

$$\sinh(ix) = i \sin x, \qquad \cosh(ix) = \cos x$$

Lemma

i) $\cosh^2 x - \sinh^2 x = 1$

ii) $\frac{d}{dx}(\cosh x) = \sinh x$

iii) $\frac{d}{dx}(\sinh x) = \cosh x$

iv) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

v) $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \sinh^{-1}(x/a) + C$

vi) $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \cosh^{-1}(x/a) + C$

vii) $\ldots$